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Abstract

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METHOD OF INTEGRAL MOMENT KINETIC EQUATIONS

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1. The apparatus of distribution functions has proved very suitable for describing nonequilibrium processes and has found wide application in the formulation of kinetic problems. However, in solving these problems the question arises: is it expedient to construct the complete solution at the level of distribution functions?

Even in the comparatively simple case of a rarefied monatomic gas, where there is a well-posed formulation of the problem in the form of an integral equation for the one-particle distribution function $f(\mathbf{r}, \mathbf{u}, t)$,

$$\dot{f} = Vf, \tag{1}$$

the following considerations lead to this question:

1) The performance of the multiple quadratures contained in the operator V, with intermediate nonlinear operations, is an extremely difficult task even for modern high-speed computers, since one has to work in the 7-dimensional space $x, y, z, u_x, u_y, u_z, t$. Therefore the question arises of a preliminary representation of a function of many variables through combinations of functions of a smaller number of variables ([3]).

2) The distribution function is invoked, as a rule, not because it is precisely it that one needs to know, but in order to attain a closed formulation of the problems. From the standpoint of the result, however, usually only certain moments of the distribution function are of interest. Therefore the question arises of finding such methods for solving kinetic problems that would lead to the required quantities by a more or less direct route, without the necessity of constructing the entire distribution function with the same accuracy. The possible lesser generality of such methods does not eliminate the need for them.

3) The operator V contains the internal transformant T and the boundary transformant $\widetilde{T}$, which describe the results of collisions of molecules with one another and with the boundaries. These transformants are, as it were, those channels through which the influence on f of a deeper level—the

mechanism of molecular interaction—is manifested. The limited nature of our knowledge of the deeper levels sets a limit to the reasonable accuracy of calculations for f. Moreover, even for the simplest models of the interaction potential, already for diatomic molecules the problems of constructing the transformants turn out to be extremely complex and remain unsolved to this day. Therefore the possibility of a simplified formulation of the collision problem, which it is natural to expect in "through"methods, seems essential.

In the present note we shall set forth a method for solving equation (1), based on representing f through the coefficients of its expansion in generalized Hermite polynomials $H^{(m)}_{i_1 \dots i_m}(\mathbf{v})$. This expansion is convenient because its first terms give precisely the principal macrocharacteristics of the gas. When it is substituted into the Boltzmann equation, as is known ([4,5]), differential moment equations of ever higher order are obtained; the question of boundary

conditions for which, up to the present, essentially remains unsolved.* The system of integral moment equations obtained below not only contains all boundary and initial conditions, but is also convenient for numerical solution by the natural method of successive approximations.

2. Let us write equation (1) in expanded form:

$$f(\mathbf{r}, \mathbf{u}, t) = f(\mathbf{r} - \mathbf{u}(t - t_*), \mathbf{u}, t_*)\, \Pi(\mathbf{r}, \mathbf{u}, t, t_*) +$$

$$+ \int_{t_*}^{t} \Phi(\mathbf{r} - \mathbf{u}(t - \tau), \mathbf{u}, \tau)\, \Pi(\mathbf{r}, \mathbf{u}, t, \tau)\, d\tau, \tag{2}$$

$$\Phi(\mathbf{r}, \mathbf{u}, t) = \int_{-\infty}^{\infty}\!\!\iiint |\mathbf{u}_1 - \mathbf{u}_2|\, \sigma(|\mathbf{u}_1 - \mathbf{u}_2|) f(\mathbf{r}, \mathbf{u}_1, t) f(\mathbf{r}, \mathbf{u}_2, t) \times$$

$$\times T(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u})\, d\mathbf{u}_1\, d\mathbf{u}_2, \tag{3}$$

$$\Pi(\mathbf{r}, \mathbf{u}, t, \tau) = \exp\left\{ - \int_{\tau}^{t} Q(\mathbf{r} - \mathbf{u}(t - q), \mathbf{u}, q)\, dq \right\}, \tag{4}$$

$$Q(\mathbf{r}, \mathbf{u}, t) = \int_{-\infty}^{\infty}\!\!\iint |\mathbf{u} - \mathbf{u}_1|\, \sigma(|\mathbf{u} - \mathbf{u}_1|) f(\mathbf{r}, \mathbf{u}_1, t)\, d\mathbf{u}_1; \tag{5}$$

$t_* = \max\{t_0, t_s\}$, t_0 is the initial instant, $t_s = t_s(\mathbf{r}, \mathbf{u}, t)$ is the largest root not exceeding t of the equation $F(\mathbf{r} - \mathbf{u}(t - t_s)) = 0$, where $F(\mathbf{r}_s) = 0$ is the boundary equation; $f(\mathbf{r}, \mathbf{u}, t_0)$ is assumed given (although this is not obligatory), and on the boundary

$$f(\mathbf{r}_s, \mathbf{u}, t)|_{u_n>0} = -\int_{u_{1n}<0}\iint f(\mathbf{r}_s, \mathbf{u}_1, t)\,\frac{u_{1n}}{u_n}\,\widetilde{T}(\mathbf{r}_s, \mathbf{u}_1, \mathbf{u}, t)\,d\mathbf{u}_1, \tag{6}$$

σ is the collision cross section, T and $\widetilde{T}$ are collision transforms (a more detailed description of the concepts used may be found in ([3])). Equation (1) is obtained by substituting expressions (3)–(6) into (2).

3. Expand f in a series (see ([4,5]))

$$f(\mathbf{r}, \mathbf{u}, t) = f_0(\mathbf{r}, \mathbf{u}, t)\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{i_1,\ldots,i_m=1}^{\infty} a^{(m)}_{i_1\ldots i_m}(\mathbf{r}, t)H^{(m)}_{i_1\ldots i_m}(\mathbf{v}), \tag{7}$$

where

$$f_0(\mathbf{r}, \mathbf{u}, t) = n(\mathbf{r}, t)[h(\mathbf{r}, t)/\pi]^{3/2}\exp\{-h(\mathbf{r}, t)|\mathbf{u} - \mathbf{U}(\mathbf{r}, t)|^2\}; \tag{8}$$

$$H^{(m)}_{i_1\ldots i_m}(\mathbf{v}) = (-1)^m e^{v^2/2}\frac{\partial^m}{\partial v_{i_1}\cdots\partial v_{i_m}}e^{-v^2/2}; \tag{9}$$

$$\mathbf{v} = \mathbf{v}(\mathbf{r}, \mathbf{u}, t) = \sqrt{2h(\mathbf{r}, t)}[\mathbf{u} - \mathbf{U}(\mathbf{r}, t)]; \tag{10}$$

$$a^{(m)}_{i_1\ldots i_m}(\mathbf{r}, t) = \frac{1}{n(\mathbf{r}, t)}\int_{-\infty}^{\infty}\iint f(\mathbf{r}, \mathbf{u}, t)H^{(m)}_{i_1\ldots i_m}(\mathbf{v})\,d\mathbf{u}; \tag{11}$$

$$a^{(0)} = 1, \qquad a^{(1)}_i = 0, \qquad \sum_{i=1}^{3} a^{(2)}_{ii} = 0. \tag{12}$$

* As has recently become known to us, expansion (7) was used by B. V. Kuksenko for solving equation (2) without the boundary term for $t_0 = -\infty$. He also independently calculated the standard integrals $Q^{(m)}$ and $\Phi^{(m,n)}$ in formulas (16)–(17) for a gas of elastic spheres ([11]).

Substituting (7) into (5), (3), and (6), we find

$$Q(\mathbf{r}, \mathbf{u}, t) = \sum_{m=0}^{\infty}\frac{1}{m!}a^{(m)}(\mathbf{r}, t)Q^{(m)}(\mathbf{r}, \mathbf{u}, t); \tag{13}$$

$$\Phi(\mathbf{r}, \mathbf{u}, t) = \sum_{m,n=0}^{\infty}\frac{1}{m!}\frac{1}{n!}a^{(m)}(\mathbf{r}, t)a^{(n)}(\mathbf{r}, t)\Phi^{(m,n)}(\mathbf{r}, \mathbf{u}, t); \tag{14}$$

$$f(\mathbf{r}_s, \mathbf{u}, t)\big|_{u_n>0} = \sum_{m=0}^{\infty} \frac{1}{m!} a^{(m)}(\mathbf{r}, t) \widetilde{\Phi}^{(m)}(\mathbf{r}_s, \mathbf{u}, t), \tag{15}$$

where

$$Q^{(m)}(\mathbf{r}, \mathbf{u}, t) = \int_{-\infty}^{\infty}\!\!\int\!\!\int |\mathbf{u} - \mathbf{u}_1| \sigma(|\mathbf{u} - \mathbf{u}_1|) f_0(\mathbf{r}, \mathbf{u}_1, t) H^{(m)}(\mathbf{v}_1)\, d\mathbf{u}_1; \tag{16}$$

$$\Phi^{(m,n)}(\mathbf{r}, \mathbf{u}, t) = \int_{-\infty}^{\infty}\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int |\mathbf{u}_1 - \mathbf{u}_2| \sigma(|\mathbf{u}_1 - \mathbf{u}_2|) f_0(\mathbf{r}, \mathbf{u}_1, t) f_0(\mathbf{r}, \mathbf{u}_2, t) H^{(m)}(\mathbf{v}_1) \times$$

$$\times H^{(n)}(\mathbf{v}_2) T(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u})\, d\mathbf{u}_1\, d\mathbf{u}_2; \tag{17}$$

$$\widetilde{\Phi}^{(m)}(\mathbf{r}_s, \mathbf{u}_1, t) = -\int\!\!\int\!\!\int_{u_{1n}<0} f_0(\mathbf{r}_s, \mathbf{u}_1, t) H^{(m)}(\mathbf{v}_1) \frac{u_{1n}}{u_n} \widetilde{T}(\mathbf{r}_s, \mathbf{u}_1, \mathbf{u}, t)\, d\mathbf{u}_1. \tag{18}$$

For given σ and T, the integrals $Q^{(m)}$ and $\Phi^{(m,n)}$ are standard functions of $\mathbf{v}$ (and n, h), identical for all problems with different $F(\mathbf{r}_s)$ and $f(\mathbf{r}, \mathbf{u}, t_0)$. For a given T, the standard integrals $\widetilde{\Phi}^{(m)}(\mathbf{v})$ will depend on $\mathbf{r}_s, t$ not only through the principal moments, but also through $\widetilde{T}$.

Substitute the expansions (13)–(15) into (4) and (2). Multiplying (2) by $H^{(m)}$ and integrating with respect to $\mathbf{u}$, we obtain an infinite system of integral moment equations equivalent to (1):

$$\dot{a}^{(m)} = \frac{1}{n} \int_{-\infty}^{\infty}\!\!\int\!\!\int H^{(m)}(\mathbf{v}) V f\, d\mathbf{u} \equiv V^{(m)}(n, U_i, h, a^{(2)}, a^{(3)}, \ldots), \quad m = 0, 1, 2, \ldots. \tag{19}$$

When writing equations for the 5 principal moments n, U_i, h, the 5 equalities (12) are used. The operators $V^{(m)}$ are nonlinear and include delayed arguments. Each of them contains 5 quadratures: 3 in $\mathbf{u}$, 1 in τ (from (2)), and 1 in q (from (4)).

4. The system (19) should naturally be solved by the reduction method together with the iteration method. Restricting ourselves to moments of order N, in the general case we shall have $(N+1)(N+2)(N+3)/6$ equations. In the presence of symmetry, the number of equations and quadratures is reduced. For example, in the problem of the structure of a

shock wave, in system (19) with $N = 4$, out of 35 equations 9 remain, and the number of quadratures in each equation decreases to 4. In expanding the system, it is of course not necessary to include at once all moments of the next order.

Solving the truncated system (19) by iteration even for $N = 7$ (120 equations) is, in contrast to solving equation (1), quite a feasible problem for modern computing machines. The effectiveness of the reduction method may be hoped for in the case of not very large deviations of f from f_0. At the same time, as analogies from wave problems suggest [6], the convergence of the reduction method in system (19) may prove stronger than the convergence of the expansion (7) itself. It is also essential that the integral

iterations in (19) admit discontinuous gas-dynamic solutions as a zeroth approximation, which cannot be said of the iterations of the differential equations ([7],[8]).

5. If the molecules are smooth elastic spheres and the reflection from the boundary is diffuse, all the standard integrals (16)–(18) are taken to the end. In the general case, in (16) and (17) there remain only the last integrals with respect to the argument of the function σ. In this case, in $\Phi^{(m,n)}$ there appear functionals of the scattering function $T_\omega(\vartheta)$ ([3]) of the form

$$\omega_k = 2\pi \frac{(2k+1)!!}{(2k)!!} \int_0^\pi T_\omega(\vartheta) \sin^{2k+1} \vartheta \, d\vartheta, \qquad k = 1, 2, \ldots \tag{20}$$

In the case of spherical molecules $T_\omega = 1/4\pi$ and $\omega_k = 1$. The number of such functionals in the system truncated at moments of order N is equal to $E[N/2]$.

If the reflection deviates from diffuse reflection, then, using the expansion (7) for f up to the boundary, it is natural also to expand the function $\hat{T}/u_n$ in the same series, while truncating it reasonably earlier than (7), since at the boundary relaxation processes, generally speaking, proceed faster than inside the gas. Then in (18) there will appear precisely those functionals, depending only on $\mathbf{r}_s, t$, which it is sufficient to know instead of $\hat{T}$. It is precisely these characteristics of reflection that must be sought theoretically or experimentally.

6. The qualitative advantage of system (19) over equation (1) consists in the fact that, on its basis, a real possibility opens up for the numerical solution of a number of the most important problems of the aerodynamics of rarefied gases, problems which until now could practically not be approached.*

For a mixture of gases with internal degrees of freedom ([9]), and also in collective interaction and in complex media, the need for such methods is still more acute. In particular, along this path an important question for the phenomenological

theory of continuous media—namely, that of the determining macrocharacteristics of the medium—will be solved in a deductive manner ([10]).

7. At the present time there is no general method for finding integral characteristics of solutions without constructing the solution itself. The complexity of the problems of kinetics compels one to seek through paths, those channels existing in nature which in each concrete circle of phenomena connect the micro- and macrolevels. One attempt of this kind has been undertaken in the present note.

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* Of course, the expansion (7) will not be effective for all problems. For example, in a near-free-molecular flow, it apparently is expedient to represent f as the sum of two such expansions.

Note: Figure translations are in progress. See original paper for figures.