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On Dini Convergence in a Function Space

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On compact Hausdorff spaces X there holds the well-known Dini convergence theorem, according to which an increasing or decreasing Moore–Smith sequence of continuous real-valued functions on X, converging pointwise to a continuous function, converges uniformly to this limiting function (see ([2]), p. 53, Theorem 1). G. Hellmberg proved in ([4]) that this property characterizes compact spaces, and I. Glicksberg ([3]) proved that Dini' s theorem for ordinary sequences characterizes pseudocompact spaces. The investigations presented here are intended to supplement these results somewhat. For this purpose we define Dini convergence in a system of functions.

Definition 1. Let $S(X)$ be a family of real-valued functions on the set X. An element $f \in S(X)$ is called a **majorant** (or **minorant**) of a Moore–Smith sequence $(f_i)_{i\in I}$ from $S(X)$, if for almost all f_i with respect to the pointwise order one has $f_i \leq f$ (respectively $f \leq f_i$) (i.e., if there exists an index $i_0 \in I$ such that for all indices $i > i_0$ following i_0 and all $x \in X$ one has $f_i(x) \leq f(x)$ (respectively $f(x) \leq f_i(x)$).

Definition 2. Let $(f_i)_{i\in I}$ be a Moore–Smith sequence from $S(X)$. Moore–Smith sequences $(g_j)_{j\in J}$, $(h_k)_{k\in K}$ from $S(X)$ are called **squeezing sequences** for the sequence $(f_i)_{i\in I}$, if $(g_j)_{j\in J}$ is increasing, i.e. $g_j \geq g_{j'}$ as soon as $j > j'$, $(h_k)_{k\in K}$ is decreasing, i.e. $h_k \leq h_{k'}$ as soon as $k > k'$, and all g_j are minorants, while all h_k are majorants, of the sequence $(f_i)_{i\in I}$.

Definition 3. A Moore-Smith sequence $(f_i)_{i\in I}$ from $S(X)$ will be called **Dini-convergent in** $S(X)$ if for $(f_i)_{i\in I}$ there exist squeezing sequences in $S(X)$ converging pointwise to one and the same function.

Remark. Let $(f_i)_{i\in I}$ be a given Dini-convergent sequence from $S(X)$, and let $(g_j)_{j\in J}$, $(h_k)_{k\in K}$ be some squeezing sequences for it, converging pointwise to a function f. Then the function f depends only on $(f_i)_{i\in I}$, and not on the squeezing sequences themselves $(g_j)_{j\in J}$, $(h_k)_{k\in K}$.

We shall call this function the **Dini limit** of the sequence $(f_i)_{i\in I}$, symbolically

$$f_i \xrightarrow[D,S(X)]{} f.$$

Here the indication of $S(X)$ is important, as will be shown in Example 2. Obviously, Dini convergence implies pointwise convergence

$$f_i \xrightarrow[D,S(X)]{} f \Rightarrow f_i \xrightarrow[\pi, lm]{} f.$$

In the case of a system $S(X)$ of continuous functions on some topological space X, Dini convergence implies even uniform convergence at points.

Lemma 1. Let $S(X) \subset \mathcal{C}(X, R)$ be a subsystem of the family of all continuous real-valued functions of the topological space X. If

$$f_i \xrightarrow[D,S(X)]{} f,$$

then $(f_i)_{i \in I}$ converges uniformly at every point

$$f_i \xrightarrow[p.t.]{} f,$$

i.e. for ···

for each $\varepsilon > 0$ and $x_0 \in X$ there exists a neighborhood $U_{x_0}(\varepsilon)$ of the point x_0 such that $|f_i(x) - f(x)| < \varepsilon$ for all $x \in U_{x_0}(\varepsilon)$ and almost all $i \in I$.

In particular, f is a continuous function, i.e. $f \in \mathcal{C}(X, R)$.

Proof. Let $x_0 \in X$ and let $(g_j)_{j \in J}$, $(h_k)_{k \in K}$ be squeezing sequences for $(f_i)_{i \in I}$ from $S(X)$ converging to the function f. Then, for each fixed $\varepsilon > 0$, there exist indices $k_0 \in K$, $j_0 \in J$ such that

$$h_{k_0}(x_0) - g_{j_0}(x_0) < \varepsilon/2. \qquad (*)$$

Since h_{k_0} and g_{j_0} are continuous, one can find neighborhoods V and W of the point x_0 such that

$$g_{j_0}(V) \subset (g_{j_0}(x_0) - \varepsilon/2,\ g_{j_0}(x_0) + \varepsilon/2)$$

and

$$h_{k_0}(W) \subset (h_{k_0}(x_0) - \varepsilon/2,\ h_{k_0}(x_0) + \varepsilon/2).$$

For the neighborhood $U = V \cap W$ of the point x_0 we have

$$g_{j_0}(y) \in (g_{j_0}(x_0) - \varepsilon/2,\ g_{j_0}(x_0) + \varepsilon/2)$$

and

$$h_{k_0}(y) \in (h_{k_0}(x_0) - \varepsilon/2,\ h_{k_0}(x_0) + \varepsilon/2)$$

for $y \in U$. From $(*)$ it follows, in view of the relation

$$g_{j_0}(x_0) \le f(x_0) \le h_{k_0}(x_0),$$

that

$$f(x_0) - g_{j_0}(x_0) < \varepsilon/2, \qquad h_{k_0}(x_0) - f(x_0) < \varepsilon/2.$$

Finally,

$$f(x_0) - \varepsilon < g_{j_0}(x_0) - \varepsilon/2 < g_{j_0}(y) \le f_i(y) \le h_{k_0}(y) < \\ < h_{k_0}(x_0) + \varepsilon/2 < f(x_0) + \varepsilon$$

for all $y \in U$ and almost all $i \in I$, i.e.

$$f_i(U) \subset (f(x_0) - \varepsilon,\ f(x_0) + \varepsilon)$$

for almost all $i \in I$.

In an analogous way we obtain

$$f(U) \subset (f(x_0) - \varepsilon,\ f(x_0) + \varepsilon),$$

which completes the proof.

Example 1. Let $X = N$ be the discrete space of natural numbers. We order the set of indices $I = N \times N$ lexicographically, i.e.

$$(n, m) \succ (n', m') \iff n > n' \text{ or } (n = n' \text{ and } m > m').$$

The Moore-Smith sequence over I

$$f_{(n,m)}(x) = \begin{cases} m, & \text{for } x = n, \\ 0, & \text{for } x \neq n \end{cases}$$

converges uniformly at each point to the zero function $f \equiv 0$, although it does not converge to it in the sense of Dini convergence (since $(f_{(n,m)})_{(n,m)\in N\times N}$ has no majorant in $\mathcal{C}(X, R)$).

Example 2. For the same space $X = N$ and the family $\mathcal{C}^*(X, R)$ of all bounded functions on X, for the sequence

$$f_n(x) = \begin{cases} n, & \text{if } x = n, \\ 0, & \text{if } x \neq n \end{cases}$$

we have $f_n \xrightarrow[p.t.]{} f \equiv 0$, but not

$$f_n \xrightarrow[D,\mathcal{C}^*(X,R)]{} f,$$

although

$$f \xrightarrow[D,\mathcal{C}(X,R)]{} f;$$

the squeezing sequences are

$$g_n(x) \equiv 0, \qquad h_n(x) = \begin{cases} m, & \text{if } x = m, \text{ if } x \leq n, \\ 0, & \text{if } x > n. \end{cases}$$

Since for a compact space X, from convergence uniform at each point there follows uniform convergence, and, on the other hand, from uniform convergence $f_i \underset{p}{\rightarrow} f$ there always follows Dini convergence in

$$\mathcal{C}^*(X, R)$$

(for this it is only necessary to consider the squeezing sequences

$$f - \frac{1}{n} \quad \text{and} \quad f + \frac{1}{n}$$

), we may formulate the quoted Dini theorem symmetrically as follows.

Dini' s Theorem. *For a compact Hausdorff space X, Dini convergence and uniform convergence in the system of all real continuous functions on X are equivalent.*

We can obtain the Helly converse of Dini' s theorem from the following result on the topologization of Dini convergence.

Theorem 1. Let X be a completely regular space. Dini convergence in the family $\mathcal{C}^*(X, R)$ of all continuous real-valued bounded functions on X is a topological convergence (i.e., there exists a topology τ in $\mathcal{C}^*(X, R)$ such that $f_i \xrightarrow[D, \mathcal{C}^*(X,R)]{} f$ and $f_i \xrightarrow{\tau} f$ for any Moore-Smith sequences are equivalent) if and only if X is compact.

Corollary. (Helmberg ([4])). If, for a completely regular space, Dini convergence and uniform convergence of Moore-Smith sequences in $\mathcal{C}^*(X, R)$ are equivalent, then X is compact.

Proof of Theorem 1. The sufficiency of the compactness condition for X follows from the validity of Dini' s theorem. To prove its necessity, suppose that X is noncompact. Let βX be the Stone-Čech compactification of the space X, and let $x_0 \in \beta X \setminus X$. Denote by $\mathfrak{U}$ the system of all closed neighborhoods of the point x_0 in βX. For each neighborhood $U \in \mathfrak{U}$ and $y \in X$, $y \notin U$, and for an arbitrary natural number $\nu \in N$, choose a continuous function $\tilde{f}^{\nu}_{(U,y)}$ on βX, equal to ν on the set U, taking the value 0 at the point y, and intermediate values at the remaining points. By $f^{\nu}_{(U,y)}$ denote the restriction of the function $\tilde{f}^{\nu}_{(U,y)}$ to X. Consider the following sets of indices:

$$I(\nu) = \{i \mid i = (U_1, \dots, U_n; y_1, \dots, y_n); \quad U_k \in \mathfrak{U},\ y_k \in X, \text{ and } y_k \notin U_k\}.$$

On $I(\nu)$ define the direction:

$$i = (U_1, \dots, U_n; y_1, \dots, y_n) \succ j = (V_1, \dots, V_m; z_1, \dots, z_m)$$

$\iff$ for (V_k, z_k) from j there exist such (U_l, y_l) from i that $(V_k, z_k) = (U_l, y_l)$.

Over the sets $I(\nu)$, $\nu \in N$, define the following Moore-Smith sequences:

$$f_i^\nu = f^\nu_{(U_1,\dots,U_n;y_1,\dots,y_n)} = \inf(f^\nu_{(U_k,y_k)}) \mid (U_k, y_k) \in i.$$

Then for each sequence $(f_i^\nu)_{i\in I(\nu)}$ one has $f_i^\nu \xrightarrow[D,\mathcal{C}^*(X,R)]{} f \equiv 0$, because $(f_i^\nu)_{i\in I(\nu)}$ is a decreasing sequence converging pointwise to $f \equiv 0$. If Dini convergence were topological, then the theorem on iterated limits would have to hold (see ([6]), p. 69). In the present case the sequence $f_{(\nu,\mathfrak{I})} = f^\nu_{\mathfrak{I}(\nu)}$; $\mathfrak{I} \in \prod I(\nu) \mid \nu \in N$ $(\mathfrak{I}(\nu) = i_\nu,$ if $\mathfrak{I} = (i_1, i_2, \dots, i_\nu, \dots))$ over the index set $N \times \prod I(\nu) \mid \nu \in N$ with the order

$$(\nu, \mathfrak{I}) \succ (\mu, \mathfrak{I}') \iff \nu > \mu \text{ and } \mathfrak{I} \succ \mathfrak{I}' \text{ coordinatewise},$$

would have to be Dini-convergent in $\mathcal{C}^*(X,R)$ to f. But this is impossible, since for $f_{(\nu,\mathfrak{I})}$ in $\mathcal{C}^*(X,R)$ there exists no majorant whatsoever.

Remark. Let $\mathcal{C}_K(X,R)$ be the family of all real-valued continuous functions with compact supports (i.e., for every $f \in \mathcal{C}_K(X,R)$ there exists a compact set K_f such that $f(x) = 0$ outside K_f).

Further, let $\mathcal{C}_\infty(X,R)$ be the family of all real-valued continuous functions for which the sets $\{x \mid |f(x)| \geq \varepsilon > 0\}$ are compact for every fixed $\varepsilon > 0$.

Then the following hold:

$$1) \quad f_i \xrightarrow[D,\mathcal{C}_K(X,R)]{} f \Rightarrow f \in \mathcal{C}_K(X,R).$$

$$2) \quad f_i \xrightarrow[D,\mathcal{C}_\infty(X,R)]{} f \Rightarrow f \in \mathcal{C}_\infty(X,R).$$

$$3) \quad f_i \xrightarrow[D,\mathcal{C}_K(X,R)]{} f \Rightarrow f_i \xrightarrow{p} f.$$

The converse is true only for a compact space X.

4) If X is locally compact, then

$$f_i \xrightarrow[D,\mathcal{C}_\infty(X,R)]{} f \Rightarrow f_i \xrightarrow{p} f.$$

(True not only for a locally compact space; the converse is valid only for a compact space X.)

We shall now consider Dini convergence of ordinary sequences in $\mathcal{C}^*(X,R)$ and, in addition to Theorem 1, prove the metrizability of this convergence.

Theorem 2. *Let X be a completely regular space. Dini convergence of ordinary sequences in the family $\mathcal{C}^*(X,R)$ of all continuous real bounded functions on X is metrizable (i.e., there exists a metric ρ in $\mathcal{C}^*(X,R)$ such that $f_n \xrightarrow[D,\mathcal{C}^*(X,R)]{} f$ and $\rho(f_n, f) \to 0$ are equivalent for arbitrary ordinary sequences in $\mathcal{C}^*(X,R)$) if and only if X is pseudocompact (i.e., every real continuous function on X is bounded).*

Proof. Iseki [5] and Bagley, Connell, McKnight [1] proved that, for pseudocompact spaces, from pointwise convergence of an ordinary sequence there follows, in general, uniform convergence of this sequence. By Lemma 1 this implies the equivalence of $f_n \xrightarrow[D\mathcal{C}^*(X,R)]{} f$ and $f_n \to f$ for ordinary sequences from $\mathcal{C}^*(X,R)$. Since uniform convergence in $\mathcal{C}^*(X,R)$ is metrizable, the first part is proved.

Now suppose that X is not pseudocompact. Then there exists in X a countable locally finite family of open sets $V_1, V_2, \ldots$ with pairwise disjoint closures [3]. In each set V_n choose a point x_n and fix functions with values between 0 and ν such that

$$g_m^\nu(x) = \begin{cases} 0, & x \notin \overline{V}_m, \\ \nu, & x = x_m. \end{cases}$$

Then

$$f_n^\nu(x) = \sum_{m=n}^{\infty} g_m^\nu(x)$$

is a continuous function on X; moreover,

$$0 \le f_n^\nu(x) \le \nu, \qquad f_n^\nu(x) = \begin{cases} 0 & \text{for } x \notin \bigcup \overline{V}_m \mid m \ge n, \\ \nu & \text{for } x \in \{x_m \mid m \ge n\}. \end{cases}$$

For each fixed $\nu \in N$ there is an ordinary decreasing sequence $(f_n^\nu)_{n \in N}$ Dini-convergent to $f \equiv 0$.

If Dini convergence is metrizable, then one can choose indices

$$n_1 n_2 \le \cdots \le n_\nu \le \cdots$$

so that $f_{n_1}^1, f_{n_2}^2, \ldots, f_{n_\nu}^\nu, \ldots$ is Dini-convergent to $f \equiv 0$. But this is impossible, because for this sequence in $\mathcal{C}^*(X,R)$ there exists no majorant.

Corollary (Glicksberg [3], Iseki [5]). *Let X be a completely regular space. X is pseudocompact if and only if Dini convergence and uniform convergence of ordinary sequences in $\mathcal{C}^*(X,R)$ are equivalent.*

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Note: Figure translations are in progress. See original paper for figures.