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## Abstract

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# MATHEMATICS

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# ON THE COMPLETENESS OF THE SYSTEM $\{e^{\lambda_n z}\}$ IN A CLOSED STRIP

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Let $\lambda_n > 0$ $(n = 1, 2, ...)$, $\lambda_n \uparrow \infty$, and

$$\tau = \lim_{R\to\infty} \sum_{\lambda_i<R} \lambda_i^{-1}.$$

Carleman proved (1) that the system $\{e^{\lambda_n z}\}$ is complete in the open strip $|\operatorname{Im} z| < \pi\tau$ (any function analytic in this strip can be approximated with arbitrary accuracy on each bounded closed set $F$ of this strip by finite linear combinations of functions from the system under consideration). In the case when the limit

$$\sigma = \lim_{n\to\infty} \frac{n}{\lambda_n}$$

exists (in this case $\tau = \sigma$), the system $\{e^{\lambda_n z}\}$, $z = x + iy$, is complete ((2) and (3), p. 286) in any open curvilinear strip $\varphi(x) < y < \varphi(x) + 2\pi\sigma$, $-\infty < x < \infty$, of width $2\pi\sigma$ (in the vertical direction), and is not complete in a strip of greater width. In this note the question is the completeness of the system $\{e^{\lambda_n z}\}$ in a closed strip.

**Theorem 1.** *Let $\lambda_n > 0$, $\lambda_n \uparrow \infty$, and suppose the sequence $\{\lambda_n\}$ can be represented as the union of two subsequences: $\{\lambda_n\} = \{\lambda'_n\} + \{\lambda''_n\}$, possessing the following properties: 1) the numbers $\lambda'_n$ $(n = 1, 2, ...)$ are zeros of an entire function of exponential type $L(z)$, with $|L(iy)| \geq B \exp(A|y|)$, $B \neq 0$, $z = x + iy$, and $L(z)$ has no zeros in the right half-plane $x \geq 0$ other than $\lambda'_n$; 2) the numbers $\lambda''_n$ $(n = 1, 2, ...)$ satisfy the condition $\sum(\lambda''_n)^{-1} = \infty$. Then any function $f(z)$, analytic in the open strip $|\operatorname{Im} z| < A$ and continuous at all finite points of the closed strip $|\operatorname{Im} z| \leq A$, can be approximated with arbitrary accuracy on each bounded set $F$ belonging to the closed strip $|\operatorname{Im} z| \leq A$ by finite linear combinations of the functions of the system $\{e^{\lambda_n z}\}$.*

**Theorem 2.** *Let $\lambda_n > 0$, $\lambda_n \uparrow \infty$, and $\{\lambda_n\} = \{\lambda'_n\} + \{\lambda''_n\}$, where: 1) $\lambda'_n$ $(n = 1, 2, ...)$ are zeros of an entire function of exponential type $L(z) \not\equiv 0$ such that $|L(iy)| \leq B \exp(A|y|)$; 2) the sequence $\{\lambda''_n\}$ is either empty, finite, or infinite, and in the last case $\sum(\lambda''_n)^{-1} < \infty$. Let, further, $P_n(z)$ $(n = 1, 2, ...)$ be finite linear combinations of functions from the system $\{e^{\lambda_n z}\}$. If the sequence*

*$\{P_n(z)\}$ converges uniformly on any bounded set $F$ belonging to the closed strip $|\operatorname{Im} z| \leq A$, then it converges uniformly in every bounded domain of the plane.*

It follows, of course, from Theorem 2 that under the conditions of this theorem the system $\{e^{\lambda_n z}\}$ is not complete in the closed strip $|\operatorname{Im} z| \leq A$.

**Theorem 3.** *Let $\lambda_n > 0$, $\lambda_n \uparrow \infty$, suppose the limit*

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = \frac{A}{\pi}$$

*exists and $\{\lambda_n\} = \{\lambda_n'\} + \{\lambda_n''\}$, where $\sum (\lambda_n'')^{-1} < \infty$ and*

$$\prod_{n=1}^{\infty} \left(1 + \frac{r^2}{\lambda_{n'}^2}\right) \leq Be^{Ar}.$$

*Let $P_{n1}(z)$, $P_{n2}(z)$ be finite linear combinations of functions respectively from the systems $\{e^{\lambda_n' z}\}$, $\{e^{-\lambda_n z}\}$, and $P_n(z) = P_{n1}(z) + P_{n2}(z)$. If the sequen-*

the sequence $\{P_n(z)\}$ converges uniformly in the rectangle $|\operatorname{Im} z| \leq A$, $|\operatorname{Re} z| \leq \delta$, where $\delta > 0$, then the subsequences $\{P_{n1}(z)\}$ and $\{P_{n2}(z)\}$ converge inside the half-planes $\operatorname{Re} z < x_1$, $\operatorname{Re} z > x_2$, respectively, with $x_2 < 0 < x_1$.

Let us note the main stages of the proofs of the theorems.

For the proof of Theorem 1, put

$$\Phi_n(s) = L(\beta) q_n(\beta) \cdot \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{e^{sz}\, dz}{(\beta - z) L(z) q_n(z)}, \qquad q_n(z) = \prod_{\nu=1}^{n} \left(1 - \frac{z}{\lambda_\nu}\right); \tag{1}$$

where $\beta > 0$. The function $\Phi_n(s)$, by condition 1) of the theorem, is regular in the strip $|\operatorname{Im} s| < A$, continuous in the closed strip $|\operatorname{Im} s| \leq A$, and

$$|\Phi_n(s)| \leq C |L(\beta)|\, |q_n(\beta)|, \qquad |\operatorname{Im} s| \leq A, \tag{2}$$

where $C$ does not depend on $n$. There is a system of circles $|z| = \rho_k$, $\rho_k \uparrow \infty$, on which $|L(z)| > \exp(-K|z|)$. The same inequality (with the same $K$, we may assume) holds for large $|z|$ also on the boundary $\Gamma$ of the angle $|\arg z| < \pi/4$. In view of this, the integration along the imaginary axis in the integral (1) may be replaced by integration over the contour $\Gamma$, after which, according to the residue theorem, we obtain in the region $\sigma \leq -(\sqrt{2}K + |t| + 1)$, $s = \sigma + it$,

$$\Phi_n(s) = e^{\beta s} - \lim_{k\to\infty} \sum_{\lambda_\nu < \rho_k} a_\nu^{(n)} e^{\lambda_\nu s}, \tag{3}$$

and the convergence in the indicated region is uniform. From condition 2) of the theorem it follows that $q_n(\beta) \to 0$ as $n \to \infty$. Therefore from (2)–(3) we conclude that for every $\varepsilon > 0$ there are such $n$ and $k$ that we shall have

$$\left| e^{\beta s} - \sum_{\lambda_\nu < \rho_k} a_\nu^{(n)} e^{\lambda_\nu s} \right| < \varepsilon, \qquad |t| \le A, \quad \sigma \le \sigma_0 = -(\sqrt{2}K + A + 1). \quad (4)$$

By replacing $s$ by $s - h$, one can ensure that an inequality of the form (4) will hold in the half-strip $|t| \le A$, $\sigma \le \sigma_0$, where $\sigma_0$ is arbitrary. It remains to add that the functions $e^{\beta_n s}$, where, for example, $\beta_n = n\pi/2A$ $(n = 1, 2, ...)$, form a system complete in the strip $|\operatorname{Im} s| < 2A$ of width $4A$.

The proofs of Theorems 2 and 3 are based on the following lemma.

**Lemma.** Suppose $\lambda'_n > 0$ and $\sum(\lambda'_n)^{-1} < \infty$. For any $\beta > 0$ there exists a function $\Phi(z) \not\equiv 0$ of the form

$$\Phi(z) = \int_0^\beta \psi(t) e^{zt}\, dt, \qquad z = x + iy, \quad (5)$$

satisfying the condition

$$|\Phi(iy)| \le \frac{1}{T(|y|)}, \qquad T(r) > \frac{1}{3r} M(r), \quad r > r_0, \qquad M(r) = \prod_{\nu=1}^{\infty} \left(1 + \frac{r^2}{\lambda'^2_\nu}\right).$$

Let us prove the lemma. From the convergence of the series it follows ($^4$), see also ($^5$), p. 35, that

$$\int_1^\infty \frac{\ln M(r)}{r^2}\, dr < \infty. \quad (6)$$

Let $M(r) = \sum_{k=0}^{\infty} \frac{r^k}{m_k}$, $T(r) = \max_{k \ge 0} \frac{r^k}{m_k}$. We have $M(r) \le T(r)$. For $T(r)$ a condition of the form (6) is fulfilled. From this condition it follows, by Carleman' s fundamental theorem on quasianalytic functions (see, for example, ($^6$), p. 31), that on $[0, \beta]$ there is a function $\psi(t) \not\equiv 0$ with the properties: $\psi^{(n)}(0) = \psi^{(n)}(\beta) = 0$ $(n = 0, 1, 2, ...)$, $|\psi^{(n)}(t)| \le m_n$ $(n = 1, 2, ...)$. In the inte-

in (5) we shall take precisely this function as $\psi(t)$. We have

$$\Phi(z) = \frac{(-1)^n}{z^n} \int_0^\beta \psi^{(n)}(t) e^{zt}\, dt,$$

whence, for any $n$, we obtain

$$|\Phi(iy)| \leq m_n |y|^{-n}$$

(let $\beta < 1$) and, consequently,

$$|\Phi(iy)| \leq [T(|y|)]^{-1}.$$

The function $M(r)$ grows no faster than an entire function of order one and minimal type; therefore, for large $n$, for example, $m_n^{-1} < n^{-n}$, whence

$$M(r) \leq \sum_{n=1}^{[2r]} \frac{r^n}{m_n} + \sum_{[2r]+1}^{\infty} \left(\frac{r}{n}\right)^n < 2rT(r) + 1 < 3rT(r).$$

The lemma is proved.

We pass to the proof of Theorem 2. Put

$$L_1(z) = L(z)\Phi(z) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n'}\right),$$

where $\Phi(z)$ is the function (5). By the lemma and condition 1) of the theorem, we have

$$|L_1(iy)| = O(|y|^{-2} e^{A|y|}).$$

Therefore there exists a function $\gamma(\xi)$, analytic outside a certain rectangle

$$|\operatorname{Im} \xi| \leq A, \qquad |\operatorname{Re} \xi| \leq d,$$

with $\gamma(\infty) = 0$, continuous up to the boundary $\Gamma$ of this rectangle, and such that

$$L_1(z) = \frac{1}{2\pi i} \int_\Gamma \gamma(\xi) e^{z\xi}\, d\xi.$$

Put

$$\omega(\mu, \alpha, P_n) = e^{-\alpha\mu} \frac{1}{2\pi i} \int_\Gamma \left[ \int_0^\xi P_n(\xi - \eta + \alpha) e^{-\mu\eta}\, d\eta \right] \gamma(\xi)\, d\xi,$$

where $\alpha$ is a real parameter. It is verified directly that the quotient of the function

$$\frac{\omega(\mu, \alpha, f)}{L_1(\mu)}, \qquad f(z) = e^{\lambda z}, \qquad L_1(\lambda) = 0,$$

as a function of $\mu$, is equal to $e^{\lambda z}$ at the point $\mu = \lambda$ and is equal to zero at the points $\mu \neq \lambda$. Hence we obtain

$$P_n(z) = \frac{1}{2\pi i} \int_C \frac{\omega(\mu, \alpha, P_n)}{L_1(\mu)} e^{\mu z} \, d\mu, \tag{7}$$

where $C$ is the boundary of the angle $|\arg \mu| < \pi/4$. The singular points of the integrand can only be the points from $\{\lambda_n\}$. From the uniform convergence of $\{P_n(z+\alpha)\}$ on $\Gamma$ it follows that on $C$

$$\left| \frac{\omega(\mu, \alpha, P_n)}{L_1(\mu)} \right| < N e^{q|\mu|} |e^{-\alpha\mu}|,$$

where $N$ and $q$ do not depend on $n$. From this estimate and representation (7) it follows that $\{P_n(z)\}$ converges uniformly in every bounded domain.

Theorem 3 is proved essentially by the same scheme.

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*Note: Figure translations are in progress. See original paper for figures.*