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A. L. Krylov

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Abstract

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HYDROMECHANICS

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PROOF OF THE INSTABILITY OF A FLOW OF A VISCOUS INCOMPRESSIBLE FLUID

(Presented by Academician A. N. Kolmogorov on 22 VI 1963)

1. The following problem is called Taylor' s problem: one seeks the flow of a viscous incompressible fluid situated between coaxial rotating cylinders of infinite height with radii $r_1 < r_2$ (see ([1]), Ch. III or ([2])). For any Reynolds number this problem has a trivial solution, in which the motion of the fluid is independent of the axial and angular coordinates z and θ and takes place with angular velocity $\omega = A + B/r^2$, where A and B are constants independent of the Reynolds number. The problem of the stability of this motion with respect to infinitely small perturbations reduces to the question whether the system

$$(L - \lambda^2 - \sigma R)(L - \lambda^2)u = -2\lambda^2 R\omega v, \qquad (L - \lambda^2 - \sigma R)v = 2RAu, \qquad (*)$$

$$L = \frac{d}{dr}\frac{1}{r}\frac{d}{dr}r,$$

$$u = v = \frac{du}{dr} = 0, \qquad \text{for } r = r_i \quad (i = 1, 2)$$

($R > 0$ we shall call the Reynolds number) has an eigenvalue $\sigma(R, \lambda)$ in the right half-plane (i.e., $\operatorname{Re}\sigma > 0$); a positive answer gives instability (see ([2])). The system $(*)$ was solved numerically by Taylor (1923), who found the critical Reynolds number at which stability is lost, and later it was likewise investigated numerically by many authors (for a bibliography see ([2,3])). The purpose of the present note is a mathematically rigorous investigation of the question posed.

2. Let us reduce the system $(*)$ to a single equation by eliminating the function u; then for v we obtain:

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$$(L-\lambda^2-\sigma R)^2(L-\lambda^2)v = -4AR^2\omega v, \qquad (1_R)$$

$$v = Lv = \frac{d}{dr}(L-\lambda^2-\sigma R)v = 0 \quad \text{for } r = r_i \ (i=1,2).$$

Here $\lambda, R > 0$; $\omega = A + B/r^2$; A and B are constants. The limiting problem as $R \to \infty$, which we shall call Rayleigh' s problem, has the form

$$(L-\lambda^2)v = -\frac{4A\omega}{\sigma^2}\, v, \qquad (1_\infty)$$

$$v = 0 \quad \text{for } r = r_i \ (i=1,2)$$

and is an analogue of equation (1_R) for the stability problem of the same trivial flow for an inviscid fluid. Problem (1_∞) is a classical Sturm–Liouville problem with a discrete spectrum $\{\sigma_k^\infty\}$, situated on the axes $\operatorname{Re}\sigma = 0$ and $\operatorname{Im}\sigma = 0$. If the real spectrum contains a point $\sigma_k^\infty \neq 0$ (and hence also $-\sigma_k^\infty$), then instability occurs; if the spectrum is purely imaginary, the flow is neutrally stable.

In the present work we are interested precisely in the unstable case, characterized by the sign of the real part of the eigenvalue. Let us denote by $\{\sigma_l^R\}_{\delta,\Sigma}$, $\{\sigma_k^\infty\}_{\delta,\Sigma}$ the parts of the spectra of the problems $\{1_R\}$ and $\{1_\infty\}$ lying outside the half-strips $\operatorname{Re}\sigma < +\delta$, $|\operatorname{Im}\sigma| < \delta$ $(\delta > 0)$ and inside the circle $|\sigma| < \Sigma$ (here δ may be arbitrarily small, Σ arbitrarily large).

Theorem. For fixed δ and Σ, and for $R \geq R_0(\delta,\Sigma)$, the number of points $\{\sigma_l^R\}_{\delta,\Sigma}$ and $\{\sigma_k^\infty\}_{\delta,\Sigma}$ is the same, and the $\{\sigma_l^R\}_{\delta,\Sigma}$ can be numbered so that

$$|\sigma_k^R - \sigma_k^\infty| < \varepsilon(R),$$

where $\varepsilon(R) \to 0$ as $R \to \infty$.

In this sense one may say that the spectrum $\{\sigma_k^\infty\}$, situated outside the negative part of the real axis, is the limit, as $R \to \infty$, of the corresponding part of the spectrum $\{\sigma_k^R\}$. Hence it follows that the instability of a viscous fluid in this problem for large Reynolds numbers results from the instability of an inviscid fluid.

Let us note, however, that for any **fixed** R the spectra $\{\sigma_l^R\}$ and $\{\sigma_k^\infty\}$ behave quite differently: $\{\sigma_k^\infty\}$ has a limiting point only at $\sigma = 0$, whereas $\{\sigma_l^R\}$ only at $\sigma = \infty$, so that at first glance the theorem may seem unexpected.

3. Let us carry out the proof in detail for the model problem with constant coefficients:

$$(D^2-\sigma R)^2 D^2 v = R^2 v, \qquad 0 < x < 1, \qquad D = \frac{d}{dx},$$

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$$v = D^2 v = D(D^2 - \sigma R)v = 0 \quad \text{for } x = 0,\ x = 1; \tag{2_R}$$

$$D^2 v = -\frac{1}{\sigma^2} v,$$

$$v = 0 \quad \text{for } x = 0,\ x = 1. \tag{2_∞}$$

A fundamental system for (2_∞) is

$$v_1 = e^{i\rho x}, \qquad v_2 = e^{-i\rho x} \qquad \left(\rho = \frac{1}{\sigma}\right), \tag{3_∞}$$

and the equation for the eigenvalues is

$$\Delta_\infty(\rho) = \det \left\| \begin{matrix} v_1(0) & v_2(0) \\ v_1(1) & v_2(1) \end{matrix} \right\| = \begin{vmatrix} 1 & 1 \\ e^{i\rho} & e^{-i\rho} \end{vmatrix} = 0. \tag{4_∞}$$

A fundamental system for (2_R) has the form

$$\begin{gathered} v_1 = e^{i\rho x}[1], \qquad v_3 = e^{k_1 x}[1], \qquad v_5 = e^{k_2 x}[1], \\ v_2 = e^{-i\rho x}[1], \qquad v_4 = e^{-k_1 x}[1], \qquad v_6 = e^{-k_2 x}[1], \end{gathered} \tag{3_R}$$

where $[a] = a + o(1)$ as $R \to \infty$, with $[a]' = [a']$;

$$k_1 = \sqrt{\sigma R} + \frac{i\rho}{2}[1], \qquad k_2 = \sqrt{\sigma R} - \frac{i\rho}{2}[1],$$

and the corresponding equation for the eigenvalues is a certain determinant of the 6th order, which must vanish. Writing out this determinant $\Delta_R(\rho)$ explicitly and performing algebraic transformations on it, one easily obtains

$$\Delta_R(\rho) = \Phi(\rho, R)[\Delta_\infty(\rho)] = \Phi(\rho, R)(\Delta_\infty(\rho) + o(1)), \tag{4_R}$$

where $\Phi(\rho, R) \neq 0$ for $\rho \neq 0$. Thus the nonzero roots of $\Delta_R(\rho)$ coincide with the roots of the equation

$$\Delta_\infty(\rho) + o(1) = 0. \tag{$4'_R$}$$

Since the roots of $\Delta_\infty(\rho) = 0$ are simple, $\Delta'_\infty(\rho_k) \neq 0$, and by the theorem on an implicit function the roots of $(4'_R)$ are given by the formula

$$\rho_k^R = \rho_k^\infty + o(1), \tag{$5'_R$}$$

i.e., since $\rho \neq 0$,

$$\delta_k^R = \delta_k^\infty + o(1), \tag{5R}$$

which is precisely the assertion of the theorem.

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4. To pass from the model equation (2_R) to equation (1_R), one must use the standard technique of ordinary differential equations with a large parameter R (see, for example, ([4]), Ch. II), which makes it possible to establish for equation (1_R) the existence of a fundamental system with the following properties:

$$y_{1R} = y_{1\infty}[1], \qquad y_{3R} = e^{k_1 x}[1], \qquad y_{5R} = e^{k_2 x}[1],$$

$$y_{2R} = y_{2\infty}[1], \qquad y_{4R} = e^{-k_1 x}[1], \qquad y_{6R} = e^{-k_2 x}[1],$$

where k_1 and k_2 have their previous values, and $\{y_{1\infty}, y_{2\infty}\}$ is some fundamental system of equation (1_∞). We emphasize once again that the symbol [1] is permutable with differentiation. Without changing our arguments, we arrive at equality (5) already for the problems (1_R)–(1_∞), i.e., at the proof of our theorem. Let us note that in deriving equality (4_R) we substantially use the form of the fundamental system of equation (2_R) (or (1_R)), more precisely, the form of the “boundary-layer”solutions, namely the fact that they are found under conditions of regular behavior in the sense of Vishik–Lyusternik (see ([5])).

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REFERENCES

1. L. D. Landau, E. M. Lifshitz, *Mechanics of Continuous Media*, Moscow, 1953.
2. Ts. Ts. Lin, *Theory of Hydrodynamic Stability*, IL, 1958.
3. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Oxford, 1961.
4. M. A. Naimark, *Linear Differential Operators*, 1954.
5. M. I. Vishik, L. A. Lyusternik, UMN, 12, issue 5 (1957).

Note: Figure translations are in progress. See original paper for figures.

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