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Abstract

Full Text

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SPECIAL STRUCTURAL SPACES OF RINGS

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A class of rings* Σ is called special if:

1) every ring in Σ is a prime ring;

2) every nonzero ideal** of a ring in Σ belongs to Σ;

3) if K is a prime ring, $A \in \Sigma$ is its nonzero ideal, then $K \in \Sigma$.

By R_Σ we shall denote the special radical determined by the special class Σ (see ([2])), and by $S_\Sigma(K)$ the set of Σ-special ideals of the ring K, i.e. such ideals I that $K/I \in \Sigma$. The set $S_\Sigma(K)$ with the topology in which the closure of a subset $M \subseteq S_\Sigma(K)$ is defined as

$$\overline{M} = \{I;\ I \in S_\Sigma(K), \quad I \supseteq \bigcap_{B \in M} B\},$$

is called the Σ-special structural space of the ring K. If U is a subset of the ring K, then we define

$$Q_\Sigma(U) = \{I;\ I \in S_\Sigma(K),\ I \supseteq U\}$$

and

$$P_\Sigma(U) = \{I;\ I \in S_\Sigma(K),\ U \nsubseteq I\}.$$

Let A be an ideal of the ring K. Then, using ([2]), p. 193, and repeating the corresponding arguments of Jacobson ([4]), p. 298, it is easy to verify that the mappings $I \to I/A$ and $J \to J \cap A$, where $I \in Q_\Sigma(A)$ and $J \in P_\Sigma(A)$, are homeomorphic mappings of $Q_\Sigma(A)$ onto $S_\Sigma(K/A)$, and of $P_\Sigma(A)$ onto $S_\Sigma(A)$, respectively. In particular, the mapping $I \to I/R_\Sigma(K)$, where $I \in S_\Sigma(K)$, is a homeomorphism of $S_\Sigma(K)$ onto $S_\Sigma(K/R_\Sigma(K))$.

Theorem 1. The Σ-special structural space $S_\Sigma(K)$ of the ring K is bicompact if and only if for every ideal A in K such that K/A is an R_Σ-radical ring, there exists a finitely generated ideal $I \in A$ such that A/I is also an R_Σ-radical ring.

For the proof it suffices to apply the structure theorem of Blair and Eagan ([6]) and the simple observation that an ideal A of the ring K belongs to no ideal I from $S_\Sigma(K)$ if and only if K/A is an R_Σ-radical ring.

Corollary 1. If every ideal I of the ring K is finitely generated, then $S_\Sigma(K)$ is bicompact for any special class of rings Σ (cf. ([7]), theorem 2).

Corollary 2. If $S_\Sigma(K)$ is bicompact, then $S_\Sigma(K)$ is homeomorphic to $S_\Sigma(I)$, where I is some finitely generated ideal of the ring K and K/I is an R_Σ-radical ring (cf. ([7]), theorem 3), since $S_\Sigma(K) = P_\Sigma(I)$ and $P_\Sigma(I)$ is homeomorphic to $S_\Sigma(I)$.

Corollary 3. If K is a ring that is not mapped homomorphically onto nonzero R_Σ-radical rings, then $S_\Sigma(K)$ is bicompact if and only if K is generated as an ideal by a finite number of elements.

Proposition 1. Let Σ be a special class of rings, and R_Σ the corresponding special radical. The condition $R_\Sigma \leq R_2$ (see ([1])), where R_2 is the Brown–McCoy radical, is necessary and sufficient

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* Only associative rings are considered.

** By an "ideal," everywhere unless otherwise stated, a two-sided ideal is meant.

for the Σ-special structure space $S_\Sigma(K)$ of every ring K with 1 to be a nonempty bicompact space. The proof follows from Corollary 3 and the characterization of R_2-radical rings as rings that are not homomorphically mapped onto nonzero rings with 1.

One can verify (by analogy with ([4]), p. 302, Proposition 2) that there is a dependence between direct decompositions of the ring K into ideals and the connectedness of the space $S_\Sigma(K)$ for the case when K is an R_Σ-semisimple ring that is not homomorphically mapped onto nonzero R_Σ-radical rings.

Consider the condition on the ring K:

C. The structure of ideals L of the ring K has complements.

Remark 1. Let Σ be a special class of rings and let K be an R_Σ-semisimple ring satisfying condition C. Then the Σ-special ideals are characterized as the complements of minimal ideals of the ring K in the structure L.

The proof in ([8]), given when Σ is the class of primitive rings, is also valid in the general case.

Remark 2. If a ring K is semisimple with respect to a hereditary radical R (see ([2])) and satisfies condition C, then K is strongly R-semisimple, i.e., all nonzero homomorphic images of the ring K are also R-semisimple.

Remark 3. If a ring K is not homomorphically mapped onto nonzero R_Σ-radical rings and satisfies condition C, then every proper ideal A of the ring K is contained in some Σ-special ideal I.

Theorem 2. *If Σ is a special class of rings and K is a strongly R_Σ-semisimple ring, then the structure L of ideals of the ring K is isomorphic to the structure of open sets of the Σ-special structure space $S_\Sigma(K)$ of the ring K and, consequently, is distributive.*

If I is an ideal in K, then K/I is R_Σ-semisimple and therefore

$$I = \bigcap_{J \in Q_\Sigma(I)} J.$$

The mapping $G : I \to Q_\Sigma(I)$ is a dual isomorphism of the structure L onto the structure of closed sets of the space $S_\Sigma(K)$. The desired isomorphism is realized by the mapping

$$I \to S_\Sigma(K) \setminus G(I).$$

Corollary 4. *Theorem 2 holds if K is an R_Σ-semisimple ring satisfying condition C (cf. ([8]), Theorem 3). In this case the structure L of ideals of the ring K is a Boolean algebra.*

Theorems 3 and 4 were proved in ([8]) for the case when Σ is the class of primitive rings. The same arguments are applicable to an arbitrary special class of rings Σ.

Theorem 3. *If a ring K satisfies condition C, then for any special class of rings Σ the space $S_\Sigma(K)$ is discrete. It is compact if and only if there exists an element $x \in K$ that belongs to no Σ-special ideal of the ring K.*

Corollary 5. *A ring K is a direct sum of a finite number of simple Σ-special rings if and only if it is R_Σ-semisimple, satisfies condition C, and its Σ-special structure space $S_\Sigma(K)$ is bicompact.*

Theorem 4. *If every proper ideal of an R_Σ-semisimple ring K is contained in some Σ-special ideal and the space $S_\Sigma(K)$ is discrete, then K satisfies condition C.*

Remark 4. In an R-semisimple ring K, where R is the upper nilpotent radical (see ([2])), condition Cr (the requirement that the structure of right ideals have complements) implies condition C. Therefore Theorem 3 is true with condition C replaced by condition Cr.

If R is a special radical, then it can be determined by several different special classes Σ_α, i.e., $R_{\Sigma_\alpha} = R$, among which there exists a greatest special class Σ_0, consisting of all

primary R-semisimple rings (see (2)). In the ring K one may consider the spaces $S_{\Sigma_\alpha}(K)$, among which the largest will be $S_{\Sigma_0}(K)$, and every $S_{\Sigma_\alpha}(K)$ is everywhere a dense subset of $S_{\Sigma_0}(K)$. Fixing a certain special class Σ_α, we can characterize the class Σ_0 among the R-semisimple rings (cf. ([10])).

Theorem 5. *An R-semisimple ring K, where R is a special radical, belongs to the largest special class Σ_0 such that $R_{\Sigma_0} = R$ if and only if every proper closed subset of the space $S_{\Sigma_{\alpha_0}}(K)$ is nowhere dense, i.e. its complement is everywhere dense.*

Proof (cf. ([10]), Theorem 11). Let $K \in \Sigma_0$, $\bar{S}_1 = S_1 \neq S_{\Sigma_{\alpha_0}}(K)$, $S_2 = S_{\Sigma_{\alpha_0}}(K) \setminus S_1$. If we put

$$K_1 = \bigcap_{I \in S_1} I$$

and

$$K_2 = \bigcap_{I \in S_2} I,$$

then from $K_1 \cdot K_2 \subseteq K_1 \cap K_2 = 0$ and $K_1 \neq 0$ it follows that $K_2 = 0$, i.e. $\bar{S}_2 = S_{\Sigma_{\alpha_0}}(K)$. Now let K be an R-semisimple ring in which every proper closed subset of the space $S_{\Sigma_{\alpha_0}}(K)$ is nowhere dense, and let A be a nonzero ideal of the ring K. Put

$$S_1 = \{I,\ I \in S_{\Sigma_{\alpha_0}}(K),\ I \supset A^*\}$$

and

$$S_2 = \{I;\ I \in S_{\Sigma_{\alpha_0}}(K),\ I \supset A,\ A^* \not\subseteq I\},$$

where A^* is the annihilator of A. It is obvious that $S_1 \cap S_2 = \emptyset$, $S_1 \cup S_2 = S_{\Sigma_{\alpha_0}}(K)$, and S_1 is a closed subset. If $S_1 \neq S_{\Sigma_{\alpha_0}}(K)$, then, by assumption,

$$\bigcap_{I \in S_2} I = 0,$$

i.e. $A = 0$. Thus $S_1 = S_{\Sigma_{\alpha_0}}(K)$, and then

$$\bigcap_{I \in S_1} I = 0,$$

i.e. $A^* = 0$ for any ideal A. Consequently, the ring K is primary, i.e. $K \in \Sigma_0$.

Examples of special classes of rings are: fields, subrings of fields, rings without zero divisors, matrix rings over fields, simple rings with 1 (Brown-McCoy radical), primitive rings (Jacobson radical), subdirectly irreducible rings with idempotent core (Andrunakievich antisimple radical), primary rings without locally nilpotent ideals (Levitzki radical), all primary rings (Baer-McCoy radical).

The number of examples can be increased if one takes into account that the intersection and union of two special classes, as well as the complement of a special class to the class of all primary rings, are special classes of rings.

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Note: Figure translations are in progress. See original paper for figures.