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Abstract

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ON SOME PROPERTIES OF SOLUTIONS OF ELLIPTIC EQUATIONS

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In this paper estimates are obtained for the modulus of continuity, Harnack' s inequality, and a Liouville-type theorem for solutions of a certain class of elliptic equations admitting degeneracies. For the investigation we use the method proposed by Moser in ([1,2]). We note that related questions are studied in ([1−11]) and others.

1. The following two lemmas are consequences of embedding theorems (see ([12−14]) and others).

Lemma 1. Let, in the ball $K_\rho\{|x| \le \rho\}$, $x = (x_1, \dots, x_n)$, the functions $\mu(x) \ge \lambda(x) \ge 0$, $\mu \in L_s$, $\lambda^{-1} \in L_t$, with $1/s + 1/t \le 2/n$; let

$$\int_{K_\rho} (\lambda u_x^2 + \mu u^2)\, dx < \infty, \qquad u_x = \operatorname{grad} u.$$

Then

$$\left(\rho^{-n} \int_{K_\rho} \mu |u|^{2k}\, dx\right)^{1/k} \le c M^{1/k}(\rho) P(\rho) \left(\rho^{2-n} \int_{K_\rho} \lambda u_x^2\, dx + \rho^{-n} \int_N \mu u^2\, dx\right), \tag{1}$$

where

$$M(\rho) = \left(\rho^{-n} \int_{K_\rho} \mu^s\, dx\right)^{1/s}, \qquad P(\rho) = \left(\rho^{-n} \int_{K_\rho} \lambda^{-t}\, dx\right)^{1/t},$$

$$1 \le k = 1 + \frac{(2/n - 1/s - 1/t)}{(1 + 1/t - 2/n)} \le \frac{n}{n-2},$$

the set $N \subseteq K_\rho$, $\operatorname{mes} N \ge c_0 \rho^n$, and the constant c depends only on c_0 and n (everywhere we assume $n \ge 3$).

In what follows, $\lambda(x)$ and $\mu(x)$ are bounds for the eigenvalues of the matrix $a(x)$ (see (3), (4)). Denote by $W_2^1(a, \Omega)$ the closure of the space $C^\infty(\Omega)$ in the norm

$$\|u\| = \left(\int_\Omega [(u_x, a u_x) + \mu u^2]\, dx\right)^{1/2},$$

by $\overset{\circ}{W}{}_2^1(a,\Omega)$ the closure of the space $C_0^\infty(\Omega)$ in the norm

$$\|u\| = \left(\int_\Omega (u_x, a u_x)\, dx \right)^{1/2},$$

and by c constants depending on n.

Lemma 2. Let, in K_ρ, the function $h(x) \in L_p$, with $1/p + 1/t \le 2/n$. Then for $u(x) \in \overset{\circ}{W}{}_2^1(a, K_\rho)$ the inequality

$$\int_{K_\rho} |u^2 h|\, dx \le c\rho^2 P(\rho) \left(\rho^{-n} \int_{K_\rho} |h|^p\, dx \right)^{1/p} \int_{K_\rho} \lambda u_x^2\, dx \tag{2}$$

is valid.

We shall first consider solutions of elliptic equations of the form

$$Lu = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} = 0, \qquad b = (b_1, \dots, b_n); \tag{3}$$

$$a = \|a_{ij}\|, \qquad \lambda(x)\xi^2 \le (\xi, a\xi) \le \mu(x)\xi^2, \qquad a = a'; \tag{4}$$

everywhere in what follows we assume that in some ball K_R the functions $\lambda(x)$ and $\mu(x)$ satisfy the conditions of Lemma 1 and that $(b^2/\lambda) \in L_p(K_R)$, with $1/s + 1/t < 2/n$ (then $k > 1$ in (1)) and $1/p + 1/t \le 2/n$. Depending on whether the local behavior of a solution is being considered or a Liouville-type theorem is being established, we shall assume, with respect to $b_i(x)$, respectively, that the following condition A or B is fulfilled:

A. For $r \leqslant r_0$: 1)

$$c \sum_{i=1}^m r^2 P(r) \left[r^{-n} \int_{K_r} (b_i^2/\lambda)^p\, dx \right]^{1/p} \leqslant \frac{1}{4},$$

where $0 \leqslant m \leqslant n$;

2)

$$\sum_{i=m+1}^n \frac{\partial b_i}{\partial x_i} \geqslant 0, \qquad \sum_{i=m+1}^n b_i x_i \leqslant 0, \qquad \frac{\partial b_i}{\partial x_i} \in L_p(K_r), \quad i \geqslant m+1, \qquad \frac{1}{p} + \frac{1}{t} \leqslant \frac{2}{n}$$

(cf. (11)).

B. $1/p + 1/t = 2/n$, and for $r \geqslant R_0$: 1)

$$c r^2 \sum_{i=1}^m P(r) \left[r^{-n} \int_{K_r} (b_i^2/\lambda)^p\, dx \right]^{1/p} \leqslant \frac{1}{4},$$

where $0 \leqslant m \leqslant n$; 2) the same as in A 2).

Definition 1. A function $u(x) \in W_2^1(a,\Omega)$ is called a generalized solution of equation (3) in the domain Ω, if for every function $\varphi(x) \in \overset{0}{W}{}_2^1(a,\Omega)$

$$\int_\Omega (\varphi_x, au_x)\,dx = \int_\Omega \varphi(b,u_x)\,dx. \tag{5}$$

Definition 2. A function $v(x) \in W_2^1(a,\Omega)$ is called a generalized subsolution of equation (3) in Ω, if for every function $\varphi(x) \in \overset{0}{W}{}_2^1(a,\Omega)$, $\varphi(x) \geqslant 0$,

$$\int_\Omega (\varphi_x, av_x)\,dx \leqslant \int_\Omega \varphi(b,v_x)\,dx. \tag{6}$$

Lemma 3. Let $v(x)$ be a nonnegative subsolution in $K_{\rho+\sigma}$, and suppose that either condition A or condition B is satisfied. Then

$$\int_{K_\rho} \lambda v_x^2\,dx \leqslant \frac{c}{\sigma^2}\int_{K_{\rho+\sigma}} \mu v^2\,dx. \tag{7}$$

For the proof one must put in (6) $\varphi = \eta^4(r)v(x)$, where $\eta = 1$ for $0 \leqslant r \leqslant \rho$, $\eta = 0$ for $r \geqslant \rho+\sigma$, and $\eta(r)$ is linear for $\rho \leqslant r \leqslant \rho+\sigma$. Combining (1) for $u = v$, $N = K_\rho$, and (7), we have

$$\left(\rho^{-n}\int_{K_\rho} \mu v^{2k}\,dx\right)^{1/k} \leqslant cM^{1/k}(\rho)P(\rho)\left(1+\frac{\rho^2}{\sigma^2}\right)(\rho+\sigma)^{-n}\int_{K_{\rho+\sigma}} \mu v^2\,dx. \tag{8}$$

Since everywhere below $r \leqslant \rho < \rho+\sigma \leqslant 4r$, it follows that $M(\rho) \leqslant cM$, $P(\rho) \leqslant cP$, where $M = M(4r)$, $P = P(4r)$.

Lemma 4. Let $v(x)$ be a nonnegative subsolution in K_{4r}; then

$$\operatorname*{vrai\,max}_{K_r} v^2(x) \leqslant c' M^{1/(k-1)} P^{k/(k-1)} r^{-n} \int_{K_{2r}} \mu v^2\,dx, \qquad c' = c'(n,s,t). \tag{9}$$

Proof. Noting that $f(v) = v^m$ is a nonnegative subsolution for $m \geqslant 1$, put $\rho_\nu = r(1+2^{-\nu})$, $\nu = 0,1,\ldots$, $\sigma_\nu = \rho_{\nu-1}-\rho_\nu = r\cdot 2^{-\nu}$,

$$q_\nu = \left(\rho_\nu^{-n}\int_{K_{\rho_\nu}} \mu v^{2k^\nu}\,dx\right)^{1/k^\nu} = \left[\left(\rho_\nu^{-n}\int_{K_{\rho_\nu}} \mu\left(v^{k^{\nu-1}}\right)^{2k}dx\right)^{1/k}\right]^{1/k^{\nu-1}}.$$

By virtue of inequality (8) for $v^{k^{\nu-1}}$, $\nu \geqslant 1$,

$$q_\nu \leqslant c^{(\nu+1)/k^{\nu-1}} M^{1/k^\nu} P^{1/k^{\nu-1}} q_{\nu-1}, \qquad \nu = 1,2,\ldots,$$

whence (9) follows, if one takes into account that

$$\lim_{\nu\to\infty} q_\nu \geqslant \operatorname{vrai}\max_{K_r} v^2(x).$$

Lemma 5. Let $u(x)$ be a nonnegative solution in K_{4r}, and suppose

$$\operatorname{mes} N\{x \in K_{2r}; u(x) \geqslant 1\} \geqslant cr^n;$$

then for $x \in K_r$

$$u(x) \geqslant \gamma(n,s,t,Q) = \exp\left[-c' Q^{(2k-1)/2(k-1)}\right], \quad \text{where } Q = MP. \tag{10}$$

For the proof, consider $w = g(u+\varepsilon)$, where $g(u) = \ln^+ u^{-1}$; similarly to how Lemma 3 was proved, we establish the inequality:

$$\int_{K_{2r}} \lambda w_x^2\, dx \leq$$

$$\leq cr^{-2} \int_{K_{4r}} \mu\, dx \leq cr^{n-2} M.$$

If we apply Lemma 1 to w with $k = 1$, noting that $w(x) = 0$ for $x \in N$, we obtain the estimate

$$r^{-n} \int_{K_{2r}} \mu w^2\, dx \leq cM^2 P.$$

Since $w = g(u+\varepsilon) \geq 0$ and $g'' \geq 0$, $w(x)$ is a nonnegative subsolution. To complete the proof it remains to apply Lemma 4 to w and let ε tend to 0.

Lemma 6. *Let $u(x)$ be a solution in the ball K_{8r}. Then*

$$\operatorname*{osc}_{K_r} u \leq \eta \operatorname*{osc}_{K_{4r}} u, \qquad \text{where } \eta = 1 - \gamma/2.$$

By Lemma 4, applied to $|u|$, $u(x)$ is bounded in K_{4r}; we may assume that $M_0 = \operatorname{vrai}\max(\pm u)$ in K_{4r}. The proof of Lemma 6 follows from applying Lemma 5 to that one of the functions $1 \pm (u/M_0)$ which is ≥ 1 on a set $N \subset K_{2r}$ with $\operatorname{mes} N \geq \frac{1}{2} \operatorname{mes} K_{2r}$.

2. Estimate of the modulus of continuity of the solution.

Theorem 1. *Let $u(x)$ be a generalized solution of equation (3) in K_{r_0}, and let condition A be satisfied; let $\lambda(x) = \mu(x)/\mu_0$ and $Q(r) \leq \varkappa$ for $0 < r \leq r_0$. Then, for $r \leq r_0/2$,*

$$\operatorname*{osc}_{K_r} u \leq 8\,(r/r_0)^\alpha \operatorname*{osc}_{K_{r_0/2}} u,$$

where α depends only on $n, s, t, \varkappa$.

Theorem 1 follows easily from Lemma 6.

Remark 1. It is obvious that if the hypotheses of Theorem 1 are satisfied uniformly in an r_0-neighborhood of any point $x \in \Omega' \subset \Omega$, $d(\Omega', \partial\Omega) \geq \delta > 0$, then $u(x)$ satisfies the Hölder condition in Ω'. In particular, it suffices to require uniform ellipticity ($\mu(x) = \mu_1$, $s = t = \infty$) and $b_i \in L_q(\Omega)$, $q \geq n$ (cf. (5)); for $q = n$, condition A 1) is satisfied by virtue of the absolute continuity property of the Lebesgue integral: for $\varepsilon > 0$ there exists $r_0 > 0$ such that, for $r \leq r_0$,

$$\int_{K(x_0,r)} |b|^n \, dx \leq \varepsilon,$$

where $K(x_0, r) = \{x;\ |x - x_0| \leq r\} \subset \Omega$.

Remark 2. If $\lambda(x) \not\equiv \mu(x)/\mu_0$, but $Q(r)$ does not grow too strongly as $r \to 0$, then one can also estimate the modulus of continuity of the solution, namely: let

$$c' Q^{(2k-1)/2(k-1)} \leq \ln \psi(r)$$

(see (10)), and let $\psi(r)$ not decrease as $r \to 0$. Then, by Lemma 6,

$$\operatorname*{osc}_{K_r} u \leq \left(1 - \frac{1}{2\psi(r)}\right) \operatorname*{osc}_{K_{4r}} u;$$

it is not hard to see that from the condition

$$\int_0 \frac{dr}{r\psi(r)} = \infty$$

there follows the continuity of $u(x)$ at the origin.

3. Harnack inequality.

Theorem 2. *Let $u(x)$ be a positive solution of equation (3) in K_{3r_0}, $\lambda(x) = \mu(x)/\mu_0$, and for $x_0 \in K_{r_0}$, $0 < r \leq 2r_0$,*

$$Q(x_0, r) = \mu_0 \left(r^{-n} \int_{K(x_0,r)} \mu^s \, dx\right)^{1/s} \left(r^{-n} \int_{K(x_0,r)} \mu^{-t} \, dx\right)^{1/t} \leq \varkappa < \infty,$$

$$c\mu_0^2 r^2 \left[r^{-n} \int_{K(x_0,r)} (b^2/\mu)^p \, dx\right]^{1/p} \left[r^{-n} \int_{K(x_0,r)} \mu^{-t} \, dx\right]^{1/t} \leq \frac{1}{4}.$$

Then in $K_{r_0/2}$

$$\max u \leq c_1 \min u,$$

where c_1 depends only on $n, s, t, \varkappa$.

For Theorem 2 there is a remark analogous to Remark 1. In the case when $\mu(x) = \mu_1 = \text{const}$ and $b_i \in L_q$, $q > n$, Theorem 2 was established by L. P. Kuptsov.

4. A Liouville-type theorem

Theorem 3. Let $u(x)$ be a generalized solution in the whole space (i.e., (5) is satisfied in every ball); let condition B be satisfied and

$$\lim_{r\to\infty} Q(r) = \chi < \infty.$$

Then, for

$$a < a_0(n,s,t,\chi) = [-\ln(1-\tfrac{1}{2}\gamma(n,s,t,\chi))]/\ln 4$$

(see (10)), it follows from the inequality

$$|u(x)| \le Ar^a \quad \text{in } K_r \ (r > R_1)$$

that $u = \text{const}$.

Proof. Let $\chi_1 > \chi$,

$$[1-\tfrac{1}{2}\gamma(n,s,t,\chi_1)] = \eta_1 < 4^{-a}, \qquad Q(r) \le \chi_1 \quad \text{for } r > R_2(\chi_1) \ge \max(R_1, R_0).$$

Fix $r_0 > R_2$ and put $R = 4^m r_0$, where m is an arbitrary natural number. By Lemma 6,

$$\operatorname*{osc}_{K_{r_0}} u \le \eta_1^m \operatorname*{osc}_{K_R} u \le 2Ar_0^a(\eta_1 4^a)^m; \qquad \eta_1 4^a < 1.$$

Letting $m \to \infty$, we obtain that $u = \text{const}$ in any ball K_{r_0}.

5. Behavior of the solution in a neighborhood of an isolated singular point

Definition 3. A function

$$u(x) \in W_2^1(a, K_R \setminus K_\rho), \qquad 0 < \rho < R,$$

is called a generalized solution of equation (3) in K_R^0, where K_R^0 is the ball K_R with its center removed, if the integral identity (5) is satisfied for every

$$\varphi(x) \in \overset{0}{W}{}_2^1(a, K_R)$$

that is equal to zero in some neighborhood of the origin.

Theorem 4. Let $u(x)$ be a generalized solution in $K_{r_0}^0$, $\lambda(x) = \mu(x)/\mu_0$, condition A be satisfied, and

$$\int_{K_\rho} \mu^2\, dx \le \tilde{c}\,\rho^2, \qquad 0 < \rho \le r_0;$$

let $Q(r) \le \chi < \infty$, and for $n = 2$ let $\mu \le \mu_1$, while for $n = 3$ let $\mu \in L_3$ in K_{r_0}. Then $u(x)$ is bounded in $K_{r_0/2}$, and for $r \le r_0/2$

$$\operatorname*{osc}_{K_r} u \le 8(r/r_0)^a \operatorname*{osc}_{K_{r_0/2}} u, \qquad a = \alpha(n,s,t,\chi).$$

For $n = 2$, the example

$$a_{ij} = \delta_{ij} \ln^2 r, \qquad b_i = 0, \qquad u = 1/\ln r$$

shows that the assertion of Theorem 4 will be false if the condition $\mu(x) \le \mu_1$ is replaced by the condition $\mu(x) \in L_p$, where p is an arbitrary number.

6. Equations of a more general form

By introducing new variables (see (15)), Theorems 1 and 2 are carried over to the case of the equation

$$Lu + c(x)u = f(x),$$

and, for $f \neq 0$, for positive solutions an inequality of the form (see (8))

$$\max u \le c_1(\min u + c_2)$$

is valid. This device, applicable also to equations of higher order, is given here for the case of elliptic equations of the second order: let

$$\mathcal{L}u + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = f,$$

where

$$\mathcal{L}u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right)$$

or

$$\mathcal{L}u = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j};$$

then the function $w = yu + z$ satisfies the equation

$$\mathcal{L}w + \sum_{i=1}^{n} D_{iw} + k_0 \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + yc \frac{\partial w}{\partial y} = yf \frac{\partial w}{\partial z}, \tag{11}$$

where

$$D_{iw} = b_i \partial w/\partial x_i = y b_i \partial^2 w/\partial x_i \partial y,$$

and $k_0 > 0$ is a constant. Equation (11), in turn, can be reduced to an equation containing only highest-order derivatives.

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Note: Figure translations are in progress. See original paper for figures.