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Abstract

Full Text

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ON OPERATORS WITH COMPLETELY CONTINUOUS ITERATIONS

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In the work of P. Kuhn ([1]) a certain class Λ^+ of linear operators in a Hilbert space $\mathfrak{H}$ is considered (for the definition of the class Λ^+, see below in § 3). It turns out that the operators of this class, along with many others, possess the following remarkable property (([1]), Theorem 2.3): if $A \in \Lambda^+$ and for some natural number p the operator A^p is completely continuous, then the operator A itself has a complete system of eigenvectors $\{x_n\}_1^\infty$, corresponding to real eigenvalues with the only possible limit point 0, and moreover for every $x \in \mathfrak{H}$

$$Ax = \sum_{n=1}^{\infty} \rho_n x_n,$$

where ρ_n $(n = 1, 2, ...)$ are certain constants.

This result leaves open the question whether, in such a case, the operator A itself is necessarily completely continuous. In ([1]) there is no example refuting this supposition, nor are any ways toward its construction indicated. Such a construction is carried out by us in the present note. At the same time, the analysis of the construction carried out here raises some new problems, formulated in § 4.

1. Let, in the Hilbert space $\mathfrak{H}$, besides the $\mathfrak{H}$-metric, i.e. the scalar product (x, y) $(\|x\|^2 = (x, x);\ x, y \in \mathfrak{H})$, there be given also a certain J-metric:

$$[x, y] = (Jx, y) \qquad (x, y \in \mathfrak{H}). \tag{1}$$

Here $J = P - Q$, where P and Q are orthoprojectors in $\mathfrak{H}$, and $P + Q = I$. With respect to the J-metric, vectors $(0 \neq)x \in \mathfrak{H}$ may be **positive** $([x, x] > 0)$, **negative** $([x, x] < 0)$, and **neutral** $([x, x] = 0)$.*

Let $\mathfrak{H}$ be a separable space, and let $\{e_k^+\}_1^\infty$ and $\{e_k^-\}_1^\infty$ be $\mathfrak{H}$-orthonormal bases of the subspaces $P\mathfrak{H}$ and $Q\mathfrak{H}$, respectively $(\dim P\mathfrak{H} = \dim Q\mathfrak{H} = \infty)$:

$$(e_j^+, e_k^-) = 0; \qquad (e_j^+, e_k^+) = (e_j^-, e_k^-) = \delta_{jk} \qquad (j, k = 1, 2, ...).$$

Then their union $\{e_k^+\}_1^\infty \cup \{e_k^-\}_1^\infty$ is an orthonormal basis of the whole space $\mathfrak{H}$, which is at the same time J-orthonormal, i.e.

$$[e_j^-, e_k^+] = 0; \qquad [e_j^+, e_k^+] = -[e_j^-, e_k^-] = \delta_{kj} \qquad (j, k = 1, 2, ...). \tag{2}$$

Consider the sequence of vectors

$$f_k^+ = e_k^+ + \frac{k}{k+1} e_k^- \qquad (k = 1, 2, ...). \tag{3}$$

Since

$$c_k^2 = [f_k^+, f_k^+] = 1 - \left(\frac{k}{k+1}\right)^2 > 0 \qquad (c_k > 0;\ k = 1, 2, ...), \tag{4}$$

all these vectors are positive.

Denote by $\mathfrak{P}$ the linear span of the vectors $\{f_k^+\}_1^\infty$, and consider its J-orthogonal complement

$$\mathfrak{N} = \{g:\ g \in \mathfrak{H};\ [g, \mathfrak{P}] = 0\}.$$

* This term of J. Bognár seems to us more successful than the term used earlier, **zero vector** (cf. [2]). We note that § 6 of the work [2] sets forth the basic facts of the geometry of the space $\mathfrak{H}$ with J-metric, which served as the starting point for our constructions.

We note that, by virtue of definition (1) and the equality $\|J\| = 1$, the estimate $|[x, y]| \leqslant \|x\|\,\|y\|$ $(x, y \in \mathfrak{H})$ holds, ensuring the continuity of the form $[x, y]$ with respect to the aggregate of the variables x, y. Therefore, in particular, for any vector

$$g = \sum_{k=1}^\infty \xi_k e_k^+ + \sum_{k=1}^\infty \eta_k e_k^- \in \mathfrak{H} \qquad \left(\sum_{k=1}^\infty |\xi_k|^2 < \infty;\ \sum_{k=1}^\infty |\eta_k|^2 < \infty\right) \tag{5}$$

the relations $[g, f_k^+] = 0$ $(k = 1, 2, ...)$ imply, by (2) and (3), the equalities

$$\xi_k - \frac{k}{k+1}\eta_k = 0 \quad (k = 1, 2, ...).$$

Thus $\mathfrak{N}$ is the closure of the linear span of the vectors

$$\bar{f}_k = \frac{k}{k+1} e_k^+ + e_k^- \quad (k = 1, 2, ...).$$

It is easy to see that

$$[\bar{f}_j, \bar{f}_k] = (\bar{f}_j, \bar{f}_k) = [f_j^+, \bar{f}_k] = (f_j^+, \bar{f}_k) = 0 \quad (j \neq k;\ j, k = 1, 2, \dots),$$

and, moreover, all vectors $\{\bar{f}_k\}_1^\infty$ are negative:

$$[\bar{f}_k, \bar{f}_k] = \left(\frac{k}{k+1}\right)^2 - 1 = -c_k^2 < 0 \quad (k = 1, 2, \dots). \tag{6}$$

2. Let us proceed to the construction of the operator A. As a complete system of its eigenvectors we choose the vectors

$$x_k = \frac{1}{c_k} f_k^+; \qquad y_k = \frac{1}{c_k} \bar{f}_k \quad (k = 1, 2, \dots).$$

Obviously,

$$[x_j, x_k] = (x_j, x_k) = [y_j, y_k] = (y_j, y_k) = 0 \quad (j \neq k);$$

$$[x_k, x_k] = -[y_k, y_k] = 1 \quad (j, k = 1, 2, \dots);$$

$$\alpha_k^2 \equiv (x_k, x_k) = (y_k, y_k) = \frac{1}{c_k^2}\left[1 + \left(\frac{k}{k+1}\right)^2\right] = \frac{(k+1)^2 + k^2}{2k+1}, \tag{7}$$

$$\alpha_k > 0 \quad (k = 1, 2, \dots).$$

As the corresponding eigenvalues we choose two monotone sequences of real numbers $\{\lambda_k\}_1^\infty$ and $\{\mu_k\}_1^\infty$:

$$\lambda_1 \geqslant \lambda_2 \geqslant \dots \to 0; \qquad -\mu_1 \geqslant -\mu_2 \geqslant \dots \to 0.$$

Certain requirements will be imposed below on the character of the decrease of λ_k and $|\mu_k|$ as $k \to \infty$.

Thus,

$$Ax_k = \lambda_k x_k; \qquad Ay_k = \mu_k y_k \quad (k = 1, 2, \dots).$$

Now, for any vector $g \in \mathfrak{H}$, we set formally

$$Ag = \sum_{k=1}^\infty \lambda_k [g, x_k] x_k - \sum_{k=1}^\infty \mu_k [g, y_k] y_k. \tag{8}$$

Let the vector g have the form (5). Then

$$[g, x_k] = \frac{1}{c_k}\left[g, e_k^+ + \frac{k}{k+1}e_k^-\right] = \frac{1}{c_k}\left(\xi_k - \frac{k}{k+1}\eta_k\right),$$

$$[g, y_k] = \frac{1}{c_k}\left[g, \frac{k}{k+1}e_k^+ + e_k^-\right] = \frac{1}{c_k}\left(\frac{k}{k+1}\xi_k - \eta_k\right) \quad (k = 1, 2, ...).$$

Substituting these expressions into the series (8) and normalizing the vectors x_k and y_k:

$$u_k = \frac{x_k}{\|x_k\|} = \frac{1}{\alpha_k}x_k; \qquad v_k = \frac{y_k}{\|y_k\|} = \frac{1}{\alpha_k}y_k \quad (k = 1, 2, ...),$$

we obtain

$$Ag = \sum_{k=1}^{\infty}\lambda_k\frac{\alpha_k}{c_k}\left(\xi_k - \frac{k}{k+1}\eta_k\right)u_k - \sum_{k=1}^{\infty}\mu_k\frac{\alpha_k}{c_k}\left(\frac{k}{k+1}\xi_k - \eta_k\right)v_k.$$

Since $\{u_k\}_1^\infty$ and $\{v_k\}_1^\infty$ are $\mathfrak{H}$-orthonormal sequences, for convergence of the last series in the norm $\|\cdot\|$, for arbitrary $\{\xi_k\}_1^\infty$ and $\{\eta_k\}_1^\infty$ from the space l^2, it suffices, for example, to put

$$\lambda_k = c_k^2, \qquad \mu_k = -c_k^2 \quad (k = 1, 2, ...). \tag{9}$$

This follows directly from (7), since

$$0 < c_k\alpha_k = \left[1 + \left(\frac{k}{k+1}\right)^2\right]^{1/2} < \sqrt{2}$$

$(k = 1, 2, ...)$.

The boundedness of the operator A under this choice of $\{\lambda_k\}$ and $\{\mu_k\}$ is also easily verified directly. However, it follows also from general considerations. The form

$$[Ag, g] = \sum_{k=1}^{\infty}c_k^2|[g, x_k]|^2 + \sum_{k=1}^{\infty}c_k^2|[g, y_k]|^2 \ (< \infty) \tag{10}$$

is real (and even positive) for all $g \in \mathfrak{H}$ $(g \neq 0)$, so that A is a so-called J-self-adjoint (and even J-positive) operator in $\mathfrak{H}$. Such operators are always closed ([3]), and since A is defined everywhere in $\mathfrak{H}$, it is bounded. However, the operator A is not completely continuous.

Indeed, for example, the $\mathfrak{H}$-orthonormal sequence $\{e_k^+\}_1^\infty$ converges weakly to zero: $e_k^+ \to 0$ $(k \to \infty)$. At the same time

$$\begin{aligned} Ae_k^+ &= \lambda_k[e_k^+, x_k]x_k - \mu_k[e_k^+, y_k]y_k = c_k^2([e_k^+, x_k]x_k + [e_k^+, y_k]y_k) \\ &= e_k^+ + \frac{k}{k+1}e_k^- + \left(\frac{k}{k+1}\right)^2 e_k^+ + \frac{k}{k+1}e_k^- \\ &= \left[1 + \left(\frac{k}{k+1}\right)^2\right]e_k^+ + 2\frac{k}{k+1}e_k^- \quad (k = 1, 2, ...), \end{aligned}$$

so that

$$\lim_{k\to\infty} \|Ae_k^+\| = 2\sqrt{2} \neq 0.$$

On the other hand, for any natural $p \geq 2$

$$\begin{aligned} A^p g &= \sum_{k=1}^{\infty} \lambda_k^p [g, x_k] x_k - \sum_{k=1}^{\infty} \mu_k^p [g, y_k] y_k \\ &= \sum_{k=1}^{\infty} c_k^{2p} \alpha_k^2 [g, u_k] u_k - (-1)^p \sum_{k=1}^{\infty} c_k^{2p} \alpha_k^2 [g, v_k] v_k. \end{aligned}$$

Therefore (see (7))

$$\left\| A^p g - \sum_{k=1}^{n} c_k^{2p} \alpha_k^2 [g, u_k] u_k + (-1)^p \sum_{k=1}^{n} c_k^{2p} \alpha_k^2 [g, v_k] v_k \right\| \leq$$

$$\leq 2 \left(\left\| \sum_{n+1}^{\infty} c_k^{2(p-1)} [g, u_k] u_k \right\| + \left\| \sum_{n+1}^{\infty} c_k^{2(p-1)} [g, v_k] v_k \right\| \right) \leq 4 \left[\sum_{n+1}^{\infty} c_k^{4(p-1)} \right]^{1/2} \|g\|.$$

Since

$$c_k^2 = \frac{2k+1}{(k+1)^2},$$

for any fixed natural $p \geq 2$ the quantity

$$r_n = 4 \left[\sum_{n+1}^{\infty} c_k^{4(p-1)} \right]^{1/2}$$

can be made arbitrarily small for sufficiently large n. Consequently, the operator A is uniformly approximated by finite-dimensional operators, and therefore it is completely continuous. From the construction given it is clear that for us what was important was not the special choice (9) of the quantities $\lambda_k, |\mu_k|$, but only the order of their decrease.

3. The class Λ^+ of operators in the space $\mathfrak{H}$ is defined in ([1]) as follows. An operator $A \in \Lambda^+$ if it is defined everywhere in $\mathfrak{H}$, $[Ag, g]$ is real for all $g \in \mathfrak{H}$ (J-self-adjointness), and $[Ag, g] > 0$

for every neutral vector g^*. It is not hard to see that the operator A constructed by us in Sec. 2 belongs precisely to this class. Moreover, we have seen (see (10)) that $[Ag, g] > 0$ for all $(0 \neq) g \in \mathfrak{H}$, i.e. the operator A is J-positive. If in our construction some $\lambda_k = 0$ (or $\mu_k = 0$) occurred, then $[Ag, g] \geq 0$, but still $A \in \Lambda^+$.

In (1) it is shown (Theorem 2.2) that the property of J-nonnegativity ($[Ag, g] \geq 0$) for operators of the class Λ^+ in the case of infinite dimensionality of the subspaces $P\mathfrak{H}$ and $Q\mathfrak{H}$ follows from the condition:

α) There exists a natural number p such that the operator A^p is completely continuous.

If $A \in \Lambda^+$ and condition α) is fulfilled, it turns out ((1), Theorem 2.3) that:

β) The spectrum of A consists of real eigenvalues of finite multiplicity with the only possible limit point at zero (zero may be an eigenvalue of arbitrary multiplicity).

Denote by K_J the totality of all bounded J-self-adjoint operators in $\mathfrak{H}$, each of which satisfies conditions α) and β). In the case when $\min\{\dim P\mathfrak{H}, \dim Q\mathfrak{H}\} < \infty$, every J-self-adjoint operator A differs from some ordinary self-adjoint operator H only by a finite-dimensional summand. If, moreover, $A \in K_J$, then the bounded self-adjoint operator H^p differs from the completely continuous operator A^p likewise only by a finite-dimensional summand, and therefore H^p is completely continuous. But it follows from this that the operator H itself, and with it the operator A, is completely continuous.**

Thus, along the way the following has been established.

Theorem. The class K_J $(J = P - Q)$ contains operators different from completely continuous ones if and only if both subspaces $P\mathfrak{H}$ and $Q\mathfrak{H}$ are infinite-dimensional.

4. In conclusion we note that the operator A constructed by us in Sec. 2 essentially has the following structure: $A = \lambda_1 P_1 + \lambda_2 P_2 + \cdots$, where $P_1, P_2, \ldots$ are pairwise orthogonal "oblique" finite-dimensional projectors in $\mathfrak{H}$ (in our case one-dimensional), and $\lambda_1, \lambda_2, \ldots$ is a sequence of real numbers converging to zero. At the same time $\|P_k\| \to \infty$ $(k \to \infty)$; the decrease of $|\lambda_k|$ was chosen so that the operator A would be bounded, but not completely continuous, while some iteration A^p of it would be completely continuous.

The question arises whether a similar construction cannot be carried out for arbitrary finite-dimensional projectors (in a Hilbert and even in a Banach space). In the special case of weighted integral operators (4) there arises a purely analytic problem: what are the conditions on the kernel $K(x, s)$ under which only some iteration of the operator (and not the operator itself) is completely continuous?

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* The class Λ^+ is of interest, for example, because it contains the "weighted" integral operators considered by M. G. Krein (4)

$$\int_a^b K(x,s)\,\varphi(s)\,d\omega(s)$$

with a nonmonotone distribution function $\omega(s)$ and a so-called absolutely positive kernel $K(x,s)$. In connection with this, some subtle facts from (4) concerning such operators now receive a new geometric illumination and generalization (see on this point (1), the footnote to Lemma 2.5).

** For this observation, as well as for the formulation of the problems stated in Sec. 4, the author is indebted to valuable discussions of his results with M. G. Krein.

Note: Figure translations are in progress. See original paper for figures.