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Abstract

Full Text

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CONTINUOUS DEPENDENCE ON THE INITIAL DATA OF THE SOLUTION OF THE FIRST BOUNDARY-VALUE PROBLEM FOR A PARABOLIC EQUATION WITH NEGATIVE TIME

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In this note we consider the parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ik}(x_1, \dots, x_n)\frac{\partial u}{\partial x_k}\right] + c(x_1, \dots, x_n)u + f(t, x_1, \dots, x_n). \quad (1)$$

Let R be a cylinder in the $(n+1)$-dimensional space $(t, x_1, \dots, x_n)$, with generators parallel to the t-axis, whose upper and lower bases lie respectively in the hyperplanes $t = 0$ and $t = -T$. Let Γ be the lateral surface of the cylinder and D the domain of the n-dimensional space lying in its base. The boundary of D is assumed smooth. The following problem is considered:

$$u\big|_{t=0} = \varphi(x, \dots, x_n), \qquad u\big|_{\Gamma} = \psi(t, x_1, \dots, x_n).$$

It is known that such a problem is ill posed: a solution does not exist for every initial function $\varphi(x_1, \dots, x_n)$, and there is no continuous dependence on the initial data. However, if a solution exists, then it is unique ([1,2]). In this note it will be shown that, in the class of solutions uniformly bounded in R, there is continuous dependence on the initial function, and an estimate will be given for the change in the solution as a function of the change in the initial function and of the constant bounding the solutions.

It is assumed that the coefficients in (1) satisfy the conditions

$$a_{ik} = a_{ki}; \qquad \lambda_2^2 \sum_{i=1}^{n} \xi_i^2 \geq \sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k \geq \lambda_1^2 \sum_{i=1}^{n} \xi_i^2; \qquad -K < c \leq 0. \quad (2)$$

In the first part we shall assume that the coefficients are only measurable and, under this assumption, obtain an estimate in the L_2 metric. Here u is a generalized solution—continuous in the aggregate of the variables, possesses generalized Sobolev derivatives with respect to x_i from L_2, and satisfies the integral identity

(φ is an arbitrary continuously differentiable function vanishing on the boundary of R):

$$\iint_R \left[u\frac{\partial \varphi}{\partial t} - \sum_{i,k=1}^{n} a_{ik}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_k} + cu\varphi \right] dX\,dt = 0. \tag{3}$$

In the second part continuous differentiability of the coefficients a_{ik} will be assumed, and the estimate will then be obtained in the C metric.

1°. Introduce the following notation. For every function $f(t, x_1, \dots, x_n)$ in the cylinder R, set

$$\|f\|_t = \left(\int_D f^2(t, x_1, \dots, x_n)\, dX \right)^{1/2}.$$

Theorem 1. Let u_1 and u_2 be two solutions of equation (1) in R such that they coincide on Γ and

$$\|u_1 - u_2\|_0 < \Delta; \qquad \|u_1 - u_2\|_{-T} < M. \tag{4}$$

Then for every t, $0 > t > -T/40$, and for sufficiently small Δ, the inequality

$$\|u_1 - u_2\|_t < \frac{Q}{\sqrt{T}} \Delta^{a/\ln(0,1T)}, \tag{5}$$

holds, where $a < 0$ is an absolute constant and Q is a constant depending on $M, K, \lambda_1, \lambda_2$ and on the diameter of the domain D.

If we prove the theorem under the assumption that the coefficients are smooth, then the required estimate for nonsmooth coefficients can be obtained by passage to the limit. We shall therefore assume henceforth that the coefficients are continuously differentiable and that the solution is classical. We precede the proof of the theorem by two lemmas.

Lemma 1. Consider the equation

$$\frac{\partial u}{\partial t} + \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial u}{\partial x_k} \right) + cu = 0, \tag{1'}$$

where the coefficients satisfy the conditions listed above. Let u be a solution of equation $(1')$ in the cylinder $R' = \{0 < t_0 < t < T < 1,\ X \in D\}$, vanishing on its lateral surface Γ'. Let

$$\|u\|_{t_0} < t_0^{\beta}; \qquad \|u\|_T < M; \qquad \left\| \left[\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 \right]^{1/2} \right\|_T < M, \tag{6}$$

where $\beta > 1,\ M > 0$ are arbitrary numbers. Then for any $t,\ t_0 < t < T/2$, the inequality

$$\|u\|_t < A(4t/T)^{\beta - 1/2}, \tag{7}$$

holds, where A is a constant depending on $K, \lambda_1, \lambda_2, M$ and the diameter of the domain D.

Proof. Introduce the auxiliary function $v = u/t^{\beta}$. The function v satisfies the equation

$$\beta v + t\frac{\partial v}{\partial t} + t \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ik} \frac{\partial v}{\partial x_k} \right) + ctv = 0.$$

Multiplying by $\partial v/\partial t$ and integrating over R', we find

$$\frac{\beta}{2} \int_{R'} \frac{\partial v^2}{\partial t}\, dX\, dt + \int_{R'} t \left(\frac{\partial v}{\partial t} \right)^2 dX\, dt -$$

$$- \int_{R'} t \left[\sum_{i,k=1}^{n} a_{ik} \frac{\partial v}{\partial x_i} \frac{\partial^2 v}{\partial x_k \partial t} \right] dX\, dt + \int_{R'} \frac{ct}{2} \frac{\partial v^2}{\partial t}\, dX\, dt = 0.$$

Taking into account the condition $a_{ik} = a_{ki}$ and the zero boundary conditions, we obtain

$$- \int_{R'} t \left[\sum_{i,k=1}^{n} a_{ik} \frac{\partial v}{\partial x_i} \frac{\partial^2 v}{\partial x_k \partial t} \right] dX\, dt = -\frac{T}{2} \int_{D} \left[\sum_{i,k=1}^{n} a_{ik} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_k} \right]_{t=T} dX +$$

$$+ \frac{t_0}{2} \int_{D_0} \left[\sum_{i,k=1}^{n} a_{ik} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_k} \right]_{t=t_0} dX + \frac{1}{2} \int_{R'} \left[\sum_{i,k=1}^{n} a_{ik} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_k} \right] dX\, dt,$$

and straightforward calculations lead us to the inequality

$$\int_{R'} \left[\sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i} \right)^2 \right] dX\, dt < \frac{1}{2\lambda_1^2} \left[\beta + \frac{M^2(\lambda_2^2 + K)}{T^{2\beta - 1}} \right]$$

or

$$\int_{R'} v^2 \, dX \, dt < \frac{1}{2G\lambda_1^2} \left[\beta + \frac{M^2(\lambda_2^2 + K)}{T^{2\beta-1}} \right],$$

where G is a constant depending on the diameter of the domain D. Passing to u and noting that $\beta < (2/T)^{2\beta-1}$, we find

$$\int_{R'} \frac{u^2}{t^{2\beta}} \, dX \, dt < \frac{1 + M^2(\lambda_2^2 + K)}{2G\lambda_1^2} \left(\frac{2}{T} \right)^{2\beta-1}. \tag{8}$$

Let now t_1 be an arbitrary number satisfying the inequality $t_0 < t_1 < T/2$. Then from (8) we find

$$\int_{t_1}^{2t_1} \int_D u^2 \, dX \, dt < \frac{1 + M^2(\lambda_2^2 + K)}{G\lambda_1^2} \left(\frac{4t_1}{T} \right)^{2\beta-1} t_1.$$

Applying the mean-value theorem to the outer integral and taking into account that $d(\|u\|_t^2)/dt \geqslant 0$, we obtain for all t, $t_0 < t \leqslant T/2$,

$$\|u\|_t < A(4t/T)^{\beta-\frac{1}{2}},$$

where $A = \sqrt{[1 + M^2(\lambda_2^2 + K)]/G\lambda_1^2}$, which was to be proved.

Lemma 2. *Suppose that for equation* $(1')$ *in the cylinder* R' *we have a solution* u *which vanishes on the lateral surface* Γ'*, and*

$$\|u\|_{t_0} < t_0^\beta, \qquad \|u\|_T < M_1. \tag{9}$$

Then for all t, $t_0 < t \leqslant T/4$,

$$\|u\|_t < \frac{A}{\sqrt{T}} \left(\frac{8t}{T} \right)^{\beta-\frac{1}{2}}. \tag{10}$$

Proof. From equation (1) we find

$$\int_{T/2}^{T} \left\| \left[\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 \right]^{1/2} \right\|_t^2 dt < \frac{M_1^2}{2\lambda_1^2},$$

and by the mean-value theorem there exists T_0, $T/2 < T_0 < T$, such that

$$\left\| \left[\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 \right]^{1/2} \right\|_{T_0} < \frac{M_1}{\lambda_1 \sqrt{T}}.$$

Put $u_1 = u\sqrt{T}$. Then, by condition (9) and taking into account that $T < 1$, we obtain

$$\|u_1\|_{T_0} < \|u_1\|_T < M_1\sqrt{T} < M_1, \qquad \left\| \left[\sum_{i=1}^{n} \left(\frac{\partial u_1}{\partial x_i} \right)^2 \right]^{1/2} \right\| < \frac{M_1}{\lambda_1}.$$

Put $M = \max\{M_1, M_1/\lambda_1\}$ and, applying Lemma 1 to u_1 in the cylinder $R_0' = \{t_0 < t < T_0,\ x \in D\}$, we obtain for $t,\ t_0 < t \leqslant T_0/2$, by inequality (7),

$$\|u_1\|_t < A_1(4t/T_0)^{\beta - \frac{1}{2}}.$$

Passing from u_1 to u and noting that $T_0 > T/2$, we obtain for $t,\ t_0 < t \leqslant T/4$,

$$\|u\|_t < \frac{A}{\sqrt{T}} \left(\frac{8t}{T} \right)^{\beta - \frac{1}{2}},$$

which was to be proved.

Proof of the theorem. Put $U = u_1 - u_2$. U satisfies the homogeneous equation

$$\frac{\partial U}{\partial t} = \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ik} \frac{\partial U}{\partial x_k} \right) + cU \tag{11}$$

and the conditions

$$\|U\|_0 < \Delta; \qquad \|U\|_{-T} < M. \tag{12}$$

Extend U to positive values of t, solving the first boundary-value problem with zero boundary conditions (which is possible, since in this case the problem is posed correctly). Make the change of variable t, setting $t = t_0 - \tau$ $(t_0 > 0)$ and $V(\tau, x) = U(t_0 - \tau, x)$. Then equation (11) will pass into (1′) with respect to the function V, and conditions (12) into the conditions

$$\|V\|_{t_0} < \Delta, \qquad \|V\|_{T+t_0} < M.$$

Represent Δ in the form $\Delta = t_0^\beta$ ($\beta = \ln\Delta/\ln t_0$). We shall assume that $\Delta < t_0$. From Lemma 2 it follows that, for every τ, $t_0 < \tau < (T+t_0)/4$, the inequality

$$\|V\|_\tau < \frac{A}{\sqrt{8\tau}}\,\Delta^{\ln\left(\frac{8\tau}{T+t_0}\right)/\ln t_0}.$$

holds. Let $t_0 = 0.1\,T$ and $t_0 < \tau < T/8$. Then

$$\|V\|_\tau < \frac{Q}{\sqrt{T}}\,\Delta^{-\ln 1.1/\ln 0.1T}, \qquad \text{where } Q = \frac{A}{\sqrt{0.8}}. \tag{13}$$

Returning to U, we obtain from (13), for t, $-T/40 \le t < 0$,

$$\|U\|_t < \frac{Q}{\sqrt{T}}\,\Delta^{-\ln 1.1/\ln(0.1\,T)}$$

and, putting $-\ln 1.1 = a$, we obtain the assertion of the theorem.

2°. Let the coefficients a_{ik} be continuously differentiable and let their derivatives be bounded by the constant K.

Theorem 2. *Let u_1 and u_2 be two solutions of equation (1) in R such that they coincide on Γ and*

$$|u_1 - u_2|_{t=0} < \Delta, \qquad |u_1 - u_2|_{t=-T} < M.$$

Then, for every t, $0 > t \ge -T/40$, and for sufficiently small Δ, the inequality

$$|u_1 - u_2| < \frac{Q_1}{T^{(n+1)/2}}\,\Delta^{a/\ln 0.1T},$$

holds, where $a < 0$ is an absolute constant and Q_1 is a constant depending on $M, K, \lambda_1, \lambda_2$, and the diameter of the domain D.

In the proof one uses Pogorzelski' s theorem ([3]).

3°. In Theorems 1 and 2 it was assumed that $T \ll 1$, and the estimates were obtained in the part of the cylinder adjacent to the hyperplane $t = 0$ ($0 > t \ge -T/40$).

For a cylinder of arbitrary size, and for any point in it, estimates can be obtained by partitioning it into parts and successively applying Theorem 1 or Theorem 2 to each part.

4°. Analogous results are obtained for the equation

$$u_t = \sum a_{ik}(t,x)u_{x_i x_k} + \sum b_i(t,x)u_{x_i} + c(t,x)u + f(t,x),$$

if, in addition, one assumes continuous differentiability with respect to t of the coefficients a_{ik}.

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Note: Figure translations are in progress. See original paper for figures.