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Abstract

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HYDROMECHANICS

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ON TWO GLUING PROBLEMS

(Presented by Academician M. A. Lavrent' ev on 8 I 1963)

Problem A. Let a line L be given,

$$x = x(s), \qquad y = y(s); \qquad y(s) \to 0, \quad x(s) \to \pm\infty \quad \text{as } s \to \pm\infty,$$

where s is arc length, smooth everywhere except at the points a_1 and a_2, at which the direction of the tangent changes by a jump. We shall call a function $\psi(x, y)$, defined in the domain D bounded by L and lying above L, a **solution of gluing problem A** if this function satisfies the following requirements:

I. In the domain D there exists a Jordan curve γ, joining the points a_1 and a_2, such that in the finite part of D cut off by γ (the domain D_1), the function $\psi(x, y)$ satisfies Poisson' s equation with constant right-hand side

$$\Delta\psi(x, y) = \text{const},$$

and in the domain $D_0 = D/D_1$ the function $\psi(x, y)$ is harmonic.

II. The lines L and γ are level lines of the function $\psi(x, y)$,

$$\psi(x, y)|_L = \psi(x, y)|_\gamma = 0.$$

III. The function $\psi(x, y)$ has continuous partial derivatives of first order in the domain D, and

$$\partial\psi/\partial x \to 0, \qquad \partial\psi/\partial y \to c \neq 0 \quad \text{as } y \to +\infty.$$

The formulated problem reduces to finding a gluing line γ on which it is possible to glue, with first derivatives, solutions of the Dirichlet problems with zero boundary data for the Laplace and Poisson equations. The value of the constant on the right-hand side of Poisson' s equation is not prescribed in advance and is one of the unknowns of the problem.

The solution of the gluing problem—the function $\psi(x, y)$—may be regarded as the stream function of a flow of an ideal incompressible fluid, vortical in the domain D_1 and potential in D_0. The constant right-hand side of Poisson' s equation is then equal to the vorticity with the opposite sign, and the condition of continuity of the first derivatives of the function $\psi(x, y)$ means continuity of the velocity field. The indicated scheme of fluid motion, in which the flow that is potential in its main part near bottom irregularities is regarded as a motion with constant vorticity, was proposed by M. A. Lavrent' ev. Some properties of flows satisfying M. A. Lavrent' ev' s scheme are considered in the paper ([1]). In the papers ([1,2]) existence theorems are proved for solutions of two problems auxiliary with respect to problem A. In the paper ([1]) a theorem is proved asserting that, for a given arbitrary analytic line γ joining the points a_1 and a_2, one can always find such a line L that the solutions of the Dirichlet problems for the Laplace and Poisson equations in the domains D_0 and D_1, respectively, are glued along the line γ. In the paper of M. A. Gol' dshtik ([2]) the existence of a solution of problem B (see below) is proved in the case of a bounded domain D.

Let us write the integral equation for the gluing line γ. For a given Jordan curve γ^* with endpoints a_1, a_2, lying in D, and for a fixed right-hand side ω of Poisson' s equation, there exists a unique function $\psi^*(x, y)$, satisfying conditions I, III and vanishing on L. Denote by $F(z)$ the analytic function mapping the domain D onto the upper half-plane:

$$F(\infty) = \infty, \qquad F(a_1) = 0, \qquad \lim_{\substack{y \to +\infty \\ \xi + i\eta = \zeta = F(z)}} F'(z) = 1.$$

The function ψ^*, as a function of the variables $\xi + i\eta = \zeta = F(z)$, admits

representation

$$\psi^*(\xi_0, \eta_0) = c\eta_0 + \frac{\omega}{2\pi} \iint_{\Delta^*} g(\xi, \eta) \ln \left| \frac{\zeta_0 - \zeta}{\zeta_0 - \bar{\zeta}} \right| d\xi \, d\eta. \tag{1}$$

Here $g(\xi, \eta) = |dF^{-1}(\zeta)/d\zeta|^2$, and by Δ^* is denoted the image of the domain D_1^* cut off by γ.

For simplicity, suppose that at the points a_1 and a_2 the tangent to L turns through an angle not less than π; then $g(\xi, \eta) < M$, and from the representation (1)

Fig. 1

there follows the continuity, up to the boundary $\eta = 0$, of the first derivatives with respect to ξ and η of the solution $\psi(\xi, \eta)$ of the gluing problem. Therefore

Fig. 1

Figure 1: Fig. 1

$$\frac{\partial\psi(\xi,\eta)}{\partial\eta} + i\frac{\partial\psi(\xi,\eta)}{\partial\xi} = 0 \quad \text{for } \xi = 0. \tag{2}$$

Requiring that condition (2) also be satisfied for $\psi^*(\xi,\eta)$, we find

$$\frac{\pi}{\omega} = \frac{1}{c}\iint_{\Delta^*} g(\xi,\eta)\frac{\eta}{\xi^2+\eta^2}\,d\xi\,d\eta = \frac{1}{c}I(\Gamma^*),$$

where by Γ^* is denoted the image of γ^* in the ζ-plane. It is now clear that the solution of the gluing problem is contained in the class of functions

$$\psi^*(\xi_0,\eta_0) = c\eta_0 + \frac{c}{2I(\Gamma^*)}\iint_{\Delta^*} g(\xi,\eta)\ln\left|\frac{\zeta_0-\zeta}{\zeta_0-\bar{\zeta}}\right|d\xi\,d\eta \tag{3}$$

and a Jordan curve Γ with endpoints $0, F(a_2)$, lying in the upper half-plane, is the image of the gluing line γ if and only if

$$2I(\Gamma)\eta_0 + \iint_{\Delta} g(\xi,\eta)\ln\left|\frac{\zeta_0-\zeta}{\zeta_0-\bar{\zeta}}\right|d\xi\,d\eta = 0 \quad \text{for } \zeta_0 \in \Gamma. \tag{4}$$

The integral equation (4) is similar to the integral equation considered in the theory of equilibrium figures of a rotating fluid.

Theorem 1. *Let the line L be symmetric with respect to the straight line $x = a_0$ passing through the midpoint of the segment $[a_1, a_2]$. Suppose that in the domain Δ of the ζ-plane, bounded by the image of the gluing line γ and the segment $[0, F(a_2)]$, the condition*

$$\partial g(\xi,\eta)/\partial\xi \le 0 \quad \text{for } \xi \ge l/2. \tag{5}$$

is satisfied. Then the gluing line γ is also symmetric with respect to the straight line $x = a_0$, and $|\operatorname{grad}\psi(x,y)| \ne 0$ on the line γ, except at the endpoints a_1, a_2 and, possibly, the point of intersection of γ with the straight line $x = a_0$.

The proof of Theorem 1 is based on the integral equation (4) for the gluing line and is carried out more or less analogously to the proof of the symmetry of equilibrium figures of a rotating fluid (see ([3]), pp. 12–14). We note that in the case when the domain D is the upper half-plane, condition (5) is satisfied at all points of the domain D.

Theorem 2. *Suppose that at some point z^0 of the gluing line γ, $|\operatorname{grad}\psi| \ne 0$.*

Fig. 2

Figure 2: Fig. 2

Then the line γ is analytic in a neighborhood of the point z^0.

We briefly set forth the method of proof of this theorem. Since the line γ is a level line of the function $\psi(x,y)$ and $|\operatorname{grad}\psi| \neq 0$ at the point z^0, the line γ has a continuous tangent in a neighborhood of the point z^0. Let us place

place the origin at the point z^0 and direct the x-axis along the tangent to γ at this point. Let the line γ in a neighborhood of the origin be given by the equation $y = y(x)$. Starting from equation (4), it is not difficult to show that

$$y(x) = f[x, y(x)] + \lambda \int_{-a}^{+a} \{[y(\xi) - y(x)] \ln[(x-\xi)^2 + (y(\xi) - y(x))^2] + + 2(\xi - x) \operatorname{arctg} \frac{y(\xi) - y(x)}{\xi - x}\Big\} d\xi, \tag{6}$$

where $f(x,y)$ is an analytic function of its arguments in the square $0 \leq |x|,\ |y| < a$, and $|f(x,y)|,\ |\operatorname{grad} f(x,y)| < \varepsilon$ in the indicated square. The quantity ε can be made arbitrarily small by the choice of a. Considering (6) as a nonlinear integral equation for the unknown $y(x)$, we shall prove the analyticity of the solution of this equation that is small in norm, unique by virtue of the contraction mapping principle. Extend equation (6) to complex values ξ, x, y, replacing integration over the segment $[-a, +a]$ by integration along straight lines joining the point x (x is a complex number) with the points $-a, +a$.

Fig. 2

We apply the method of successive approximations to find a solution of the extended equation, choosing $y_0(x) = 0$ as the initial approximation. All subsequent approximations will be analytic functions in some domain containing within it the segment $(-a, +a)$ of the real axis with its endpoints removed. The solution of the extended equation, being the limit of a uniformly convergent sequence of analytic functions, is an analytic function. Since each approximation $y_k(x)$ is a real function for real x, the solution of equation (6) is also an analytic function.

The integral equation (4) for the gluing line was studied numerically by means of the M-20 electronic computer in two cases: a) the domain D is the upper half-plane; b) the domain D is the union of the upper half-plane and a disk with center at the origin.

To find the gluing line, the method of successive approximations was used. Choosing as the zeroth approximation a certain line γ_0^* passing through the points a_1 and a_2, we then found, by formula (3), the geometric locus of points at which $\psi_0^*(x, y) = 0$. The resulting curve γ_1^* was regarded as the first approximation, and so on. In both cases practical convergence of the process of successive approximations to the line γ was observed, independent of the choice of the initial approximation. The lines γ found and the pictures of the streamlines (level lines of the function $\psi(x, y)$), corresponding to a velocity at infinity equal to 1, are shown in Figs. 1 and 2. The magnitudes of the velocity at the points of intersection of γ and L with the y-axis in cases a) and b) are respectively 1.67, 1.66 and 1.39, 1.21.

Let us note that the flow characteristics obtained in case b) according to M. A. Lavrent' ev' s scheme are in qualitatively good agreement with experiment. More precisely, at points of the y-axis an increase in the velocity of the main flow is indeed observed; the pattern of streamlines in the domain D_1 and the position of the stationary point ($\operatorname{grad} \psi = 0$) differ little from those observed.

Problem B. In Problem B, in contrast to Problem A, the right-hand side of Poisson' s equation is prescribed in advance. In the case of a bounded domain D, with a piecewise-smooth boundary L, Problem B can be formulated as follows. A function $\psi(x, y)$, continuous in the closed domain D and taking on L the prescribed values $f(x, y)$, is called a solution of Problem B if in the domain D there exists a Jordan curve γ, with endpoints on L or closed, satisfying the conditions:

I. In one of the two parts into which the line γ divides the domain D, the function $\psi(x, y)$ is harmonic, while in the other it satisfies Poisson' s equation with prescribed constant right-hand side

$$\Delta\psi(x, y) = 0, \quad z \in D_0; \qquad \Delta\psi(x, y) = \omega, \quad z \in D_1. \tag{7}$$

II. In the domains D_0 and D_1 the function $\psi(x, y)$ solves the Dirichlet problems

$$\psi(x, y)\big|_\gamma = 0, \qquad \psi(x, y)\big|_L = f(x, y). \tag{8}$$

III. The first partial derivatives of the function $\psi(x, y)$ are continuous in the domain D.

We point out one easily verified variational property of Problem B. Let γ be a smooth Jordan curve dividing the domain D into two parts D_0 and D_1, and let $\psi(x, y)$ be a function solving the Dirichlet problems (7), (8) for these domains. Put

$$\Phi(\gamma) = \iint_{D_0} (\psi_x^2 + \psi_y^2)\, dx\, dy + \iint_{D_1} (\psi_x^2 + \psi_y^2 + 2\omega\psi)\, dx\, dy.$$

Then the function $\psi(x, y)$ is a solution of Problem B if and only if the line γ gives an extremum to the functional $\Phi(\gamma)$.

Below we give a result concerning the number of solutions of Problem B in the simplest case, when the domain D is a half-plane and $f(x, 0) = -Q/2 = \text{const}$.

In this case, to the conditions I–III listed above one must add the conditions at infinity

$$\partial\psi/\partial x \to 0, \qquad \partial\psi/\partial y \to 1 \quad \text{as } y \to +\infty.$$

In addition, we impose an additional requirement on the gluing line γ:

$$\{x = x(t),\ y = y(t)\}$$

$$0 < y(t) < M, \qquad x(t) \to \pm\infty \quad \text{as } t \to \pm\infty. \tag{9}$$

In this formulation, Problem B for the half-plane is equivalent to the following integral equation for the gluing line:

$$y_0 + \frac{\omega}{2\pi} \iint_{D_1} \ln\left|\frac{z_0 - z}{\bar{z}_0 - z}\right| dx\,dy = -\frac{Q}{2} \quad \text{for } z_0 \in \gamma. \tag{10}$$

We shall agree to understand by $\gamma \in R$ a line satisfying condition (9).

Theorem 3. For $\omega < 0$, $Q \leq 0$ and $\omega > 0$, $\omega Q > 1$, there is no solution of Problem B for the half-plane with gluing line $\gamma \in R$. For $Q > 0$, $\omega Q \leq 0$, there exists only one solution of the problem with gluing line $\gamma \in R$. For $Q > 0$, $0 < \omega Q < 1$, there exist at least two solutions of the problem with gluing line $\gamma \in R$, and there exists only one solution if one additionally requires that

$$\partial\psi/\partial y\big|_{y=0} \geq \sqrt{1 - \omega Q}.$$

The proof of Theorem 3 is based on the integral equation (10) and is carried out with the aid of suitably chosen barrier functions of the form $\varphi(y) = \alpha y^2 + \beta y$.

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Note: Figure translations are in progress. See original paper for figures.