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Abstract

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COMPLETE CONTINUITY AND STRONG CONTINUITY OF URYSOHN INTEGRAL OPERATORS

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It is well known that Urysohn integral equations and Urysohn integral operators play an important role in nonlinear analysis. In studying their properties it is often necessary to consider such function spaces in which these operators act and are continuous or completely continuous. It is especially often required that they be completely continuous. Therefore the question of conditions for complete continuity of a Urysohn operator is important. In papers ([1−3]) this problem was considered under various conditions in the spaces C, L^p, and, in particular, in the Orlicz spaces L_M^*. L. A. Ladyzhenskii in paper ([2]) gave rather broad sufficient conditions for complete continuity of a Urysohn operator in the space C. In the present article the author continues the study of this question.

1. Let G be a bounded closed subset of a finite-dimensional Euclidean space, and let C be the space of all continuous functions on G. Let $C = C + C + \dots + C$, i.e. C is the space of continuous vector-functions, whose norm is defined by the formula

$$\|\varphi\| = \max_i \|\varphi_i\|,$$

where $\varphi(t) \in (\varphi_1(t), \dots, \varphi_m(t)) \in C$.

It is assumed that the functions $K_j(s, t, u_1, \dots, u_m)$ $(j = 1, 2, \dots, m)$ are defined for $s, t \in G$, $-\infty < u_i < \infty$, and satisfy the conditions:

I_1. For each $s \in G$ the functions $K_j(s, t, u_1, \dots, u_m)$ are continuous in $u_1, \dots, u_m$ for almost all $t \in G$ and are measurable in t for each s and each $(u_1, \dots, u_m)$.

The operator

$$\mathcal{K}\varphi(s) = (\mathcal{K}_1\varphi(s), \dots, \mathcal{K}_m\varphi(s)), \tag{1}$$

where

$$\mathcal{K}_j\varphi(s) = \int_G K_j[s,t,\varphi_1(t),\dots,\varphi_m(t)]\,dt \quad (j = 1,\dots,m),$$

will be called a **Urysohn operator**.

It is easy to see that, in order to prove complete continuity of the operator $\mathcal{K}$ in C, it suffices to consider only one $\mathcal{K}_j$, i.e. an operator acting from C into C. Below only an operator of the form

$$\mathcal{K}\varphi(s) = \int_G K[s,t,\varphi_1(t),\dots,\varphi_m(t)]\,dt$$

will be considered.

2. Suppose that

$$\mathrm{I}_2. \quad K(s,t,-u_1,u_2,\dots,u_m) = -K(s,t,u_1,u_2,\dots,u_m)$$

for

$$-\infty < u_1,\dots,u_m < \infty.$$

Lemma 1. Suppose that conditions I_1 and I_2 are satisfied. If for every $\varphi \in C$, $\|\varphi\| \le \alpha$, where α is some positive number,

$$\mathcal{K}\varphi(s) = \int_G K\{s,t,\varphi_1(t),\dots,\varphi_m(t)\}\,dt < \infty, \qquad s \in G,$$

then for every $s \in G$ the inequality

$$\int_G \sup_{\substack{|u_i|\le\alpha \\ (i=1,\dots,m)}} |K[s,t,u_1,\dots,u_m]|\,dt < \infty$$

holds.

With the aid of this lemma one proves

Theorem 1. Suppose that conditions I_1, I_2 are satisfied and the operator $\mathcal{K}$ acts from C into C. If $\mathcal{K}$ is bounded, then it is weakly continuous and the function $K(t,s,u_1,\dots,u_m)$ satisfies the condition:

II_1^*. For every $\alpha > 0$ there exists a $\beta > 0$ such that

$$\int_G \sup_{\substack{|u_i|\le\alpha \\ j=1,\dots,m}} |K(s,t,u_1,\dots,u_m)|\,dt \le \beta, \qquad s \in G.$$

If I_1, I_2 are satisfied, then from the compactness of the operator $\mathcal{K}$, acting from C into C, there follows the complete continuity and the strengthened continuity of $\mathcal{K}$.

In the following lemma we consider properties of the function $K(s, t, u_1, \dots, u_m)$ when the operator $\mathcal{K}$ is compact.

Lemma 2. If conditions I_1 and I_2 are satisfied, the operator $\mathcal{K}$ acts from C into C and is compact, then it satisfies condition II_1 and the following condition:

II_2^{**}. For every $\varepsilon > 0$ and every $s \in G$ there exists a $\delta > 0$ such that

$$\int_G \sup |K(s+h, t, u_1, \dots, u_m) - K(s, t, u_1, \dots, u_m)| \, dt < \varepsilon \qquad \text{for } \|h\| < \delta.$$

From Theorem 1 and Lemma 2 there immediately follows

Theorem 2. If I_1, I_2 are satisfied, then the necessary and sufficient condition for compactness of the operator $\mathcal{K}$ is that both conditions II_1, II_2 be satisfied.

Remark. It is easy to see that condition II_2 may be replaced by a formally stronger one; that is, if $\mathcal{K}$ is compact, then $K(s, t, u_1, \dots, u_m)$ satisfies II_1 and

$$II_2'. \quad \lim_{\|h\| \to 0} \sup_{s, s+h \in G} \int_G \sup |K(s+h, t, u_1, \dots, u_m) - K(s, t, u_1, \dots, u_m)| \, dt = 0.$$

3. Suppose that $K(s, t, u_1, \dots, u_m)$ satisfies I_1 and

$$I_3. \quad K(s, t, 0, u_2, \dots, u_m) \equiv 0.$$

Putting

$$K^{(1)}(s, t, u_1, \dots, u_m) = \begin{cases} K(s, t, u_1, \dots, u_m), & \text{if } u_1 \geq 0, \\ -K(s, t, -u_1, \dots, u_m), & \text{if } u_1 \leq 0, \end{cases}$$

we easily verify that the function $K^{(1)}(s, t, u_1, \dots, u_m)$ also satisfies condition I_2.

* Condition II_1 was introduced by L. A. Ladyzhenskii in ([2]) for $m = 1$.

** Condition II_2 was also given by L. A. Ladyzhenskii in ([2]) for $m = 1$.

Let $\mathcal{K}^{(1)}$ be the operator defined by the function $K^{(1)}(s,t,u_1,\dots,u_m)$. For each $\varphi(t)\in C$ put

$$\varphi_1^{(+)}(t)=\max\{\varphi_1(t),0\},\qquad \varphi_1^{(-)}(t)=\max\{-\varphi_1(t),0\},$$

$$\varphi^{(+)}(t)=(\varphi_1^{(+)}(t),\varphi_2(t),\dots,\varphi_m(t)),\qquad \varphi^{(-)}(t)=(\varphi_1^{(-)}(t),\varphi_2(t),\dots,\varphi_m(t)).$$

Then we have

$$\begin{aligned}\mathcal{K}^{(1)}\varphi(s)&=\int_G K^{(1)}[s,t,\varphi_1(t),\dots,\varphi_m(t)]\,dt\\&=\int_G K[s,t,\varphi_1^{(+)}(t),\varphi_2(t),\dots,\varphi_m(t)]\,dt\\&\quad-\int_G K[s,t,\varphi_1^{(-)}(t),\varphi_2(t),\dots,\varphi_m(t)]\,dt\\&=\mathcal{K}\varphi^{(+)}(s)-\mathcal{K}\varphi^{(-)}(s).\end{aligned}$$

Therefore, from the boundedness and compactness of $\mathcal{K}$ there follow, respectively, the boundedness and compactness of $\mathcal{K}^{(1)}$; consequently, if the operator $\mathcal{K}$ is bounded, then the function $K^{(1)}(s,t,u_1,\dots,u_m)$ satisfies Π_1, and therefore the function $K(s,t,u_1,\dots,u_m)$ for $0\le u_1\le\alpha,\ |u_i|\le\alpha\ (i=2,\dots,m)$ also satisfies condition Π_1; if $\mathcal{K}$ is compact, then $K^{(1)}(s,t,u_1,\dots,u_m)$ satisfies Π_1,Π_2, and therefore the function $K(s,t,u_1,\dots,u_m)$ for $0\le u_1\le\alpha,\ |u_i|\le\alpha\ (i=2,\dots,m)$ also satisfies conditions Π_1,Π_2. In a similar way one can prove that from the boundedness of the operator $\mathcal{K}$ there follows condition Π_1 for $-\alpha\le u_1\le 0,\ |u_i|\le\alpha\ (i=2,\dots,m)$, and from the compactness of $\mathcal{K}$ there follow conditions Π_1,Π_2 for $-\alpha\le u_1\le 0,\ |u_i|\le\alpha\ (i=2,\dots,m)$.

Now suppose that the function $K(s,t,u_1,\dots,u_m)$ is arbitrary, but satisfies I_1. Putting

$$K_1(s,t;u_1,\dots,u_m)=K(s,t;u_1,\dots,u_m)-K(s,t;0,u_2,\dots,u_m),$$

$$K_2(s,t;u_2,\dots,u_m)=K(s,t;0,u_2,\dots,u_m)-K(s,t;0,0,u_3,\dots,u_m),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$K_m(s,t;\dots;u_m)=K(s,t;0,\dots,0,u_m)-K(s,t;0,\dots,0),$$

we obtain that

$$K(s,t,u_1,\dots,u_m)=\sum_{j=1}^{m}K_j(s,t,u_j,\dots,u_m)+K(s,t;0,\dots,0).$$

It is obvious that each function $K_j(s,t;u_j,\dots,u_m)$ satisfies not only I_1, but also I_3; therefore the following holds.

Theorem 3. If the operator $\mathcal{K}$ acts from C into C and is bounded, then it is weakly continuous and satisfies condition Π_1.

Theorem 4. If the operator $\mathcal{K}$ acts from C into C, then a necessary and sufficient condition for the compactness of $\mathcal{K}$ is the satisfaction of both conditions Π_1,Π_2. From the compactness of the operator $\mathcal{K}$ there follows its complete and strengthened continuity.

4. Finally, let us consider the question of complete continuity of the Hammerstein integral operator. Suppose that $K(s,t)$ is defined for $s,t\in G$, and $f(t,u_1,\dots,u_m)$ is defined for $t\in G$, $-\infty<u_i<\infty$ $(i=1,\dots,m)$ and satisfies the Carathéodory conditions (see (3)). Put

$$a_\alpha(t)=\sup_{\substack{|u_i|\le\alpha\\ i=1,\dots,m}} f(t,u_1,\dots,u_m).$$

Theorem 5. If for every $\alpha>0$ there exist positive numbers $a_1<a_2$ such that

$$a_1\leqslant a_\alpha(t)\leqslant a_2, \tag{2}$$

then a necessary and sufficient condition for the complete continuity of the operator

$$\mathfrak{K}f\varphi(s)=\int_G K(s,t)\,f[t,\varphi_1(t),\dots,\varphi_m(t)]\,dt$$

is the fulfillment of the following two conditions:

$\overline{\Pi}_1$. For every $s\in G$

$$\int_G |K(s,t)|\,dt<\infty.$$

$\overline{\Pi}_2$. For every $\varepsilon>0$ there exists $\delta>0$ such that

$$\int_G |K(s+h,t) - K(s,t)|\,dt < \varepsilon$$

for every $s \in G$, whenever $\|h\| < \delta$.

If $f(t,u) = u$, then the operator Kf becomes linear and $a_\alpha(t) = \alpha$; thus condition (2) is naturally fulfilled. Consequently, a necessary and sufficient condition for the complete continuity of the operator

$$\mathfrak{K}\varphi(s) = \int_G K(s,t)\varphi(t)\,dt$$

is the fulfillment of the conditions $\overline{\Pi}_1, \overline{\Pi}_2$.

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Note: Figure translations are in progress. See original paper for figures.