Mathematics

G. V. VIRABYAN

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Abstract

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G. V. VIRABYAN

ON THE SPECTRAL PROPERTIES OF OPERATORS GENERATED BY SYSTEMS OF DIFFERENTIAL EQUATIONS OF HIGHER ORDER OF S. L. SOBOLEV TYPE

(Presented by Academician S. L. Sobolev on 29 XI 1962)

In the present note we give certain results, part of which are a development of the corresponding results from ([1–7]).

1. Let Ω be a bounded domain of n-dimensional Euclidean space, bounded by a sufficiently smooth surface Γ. Consider the Hilbert space H_ν, obtained from the linear manifold D_ν of solenoidal vectors smooth in the closed domain $\overline{\Omega}$ ($\operatorname{div}_\nu \mathbf{v} = 0$) by completion with respect to the scalar product

$$(\mathbf{v}^{(1)}, \mathbf{v}^{(2)})_\nu = \int_\Omega \sum_{\substack{\beta_1+\dots+\beta_n=\nu \\ \alpha_1+\dots+\alpha_n=\nu}} \left\{b^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_n}\right\}^{-1} v^{(1)}_{\alpha_1,\dots,\alpha_n} v^{(2)}_{\beta_1,\dots,\beta_n}\, d\Omega, \tag{1}$$

where $\left\{b^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_n}\right\}^{-1}$ is the matrix inverse to the positive definite matrix

$$B = \left\|b^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_n}\right\|.$$

Here and in what follows the summation extends over all possible nonrepeating systems $(\alpha_1, \dots, \alpha_n)$ for which $\alpha_1+\dots+\alpha_n = \nu$, and each of the α_i $(i = 1, 2, \dots, n)$ takes integer values from 0 to ν.

In H_ν consider the operator Π_ν, defined by the formula $\Pi_\nu \mathbf{v} = \mathbf{w},\ \mathbf{v} \in D_\nu$;

$$w_{\alpha_1,\dots,\alpha_n} = \sum_{\beta_1+\dots+\beta_n=\nu} \left\{ a^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_n} v_{\beta_1,\dots,\beta_n} + b^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_n} \frac{\partial^\nu P}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \right\} \tag{2}$$

$$(\alpha_1 + \dots + \alpha_n = \nu),$$

where P is determined from the boundary-value problem

$$L_\nu(P) = -\sum_{\substack{\alpha_1+\dots+\alpha_n=\nu \\ \beta_1+\dots+\beta_n=\nu}} \frac{\partial^\nu}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \left(b^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_n} \frac{\partial^\nu P}{\partial x_1^{\beta_1}\cdots\partial x_n^{\beta_n}} \right) =$$

$$= \sum_{\substack{\alpha_1+\dots+\alpha_n=\nu \\ \beta_1+\dots+\beta_n=\nu}} \frac{\partial^\nu}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \left(a^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_n} v_{\beta_1,\dots,\beta_n} \right), \tag{3}$$

$$P\Big|_\Gamma = \frac{\partial P}{\partial n}\Big|_\Gamma = \dots = \frac{\partial^{\nu-1} P}{\partial n^{\nu-1}}\Big|_\Gamma = 0. \tag{4}$$

L_ν is an elliptic operator of order 2ν; n is the outward normal to the boundary Γ. With respect to the matrices

$$A = \left\| a^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_n} \right\| \quad \text{and} \quad B = \left\| b^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_n} \right\|$$

we assume that they are real, symmetric, mutually commuting constant matrices, with $A^2 = E$, and B positive definite.

The operator Π_ν under consideration is generated by the following class of systems of differential equations of S. L. Sobolev type:

$$\frac{\partial v_{\alpha_1,\dots,\alpha_n}}{\partial t} = \sum_{\beta_1+\dots+\beta_n=\nu} \left\{ a^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_n} v_{\beta_1,\dots,\beta_n} + b^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_n} \frac{\partial^\nu P}{\partial x_1^{\beta_1}\cdots\partial x_n^{\beta_n}} \right\}$$

$$(\alpha_1 + \dots + \alpha_n = \nu), \tag{5}$$

$$\operatorname{div}_\nu \mathbf{v} \equiv \sum_{\alpha_1+\dots+\alpha_n=\nu} \frac{\partial^\nu v_{\alpha_1,\dots,\alpha_n}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} = 0.$$

Let us also consider the Hilbert space H^ν, which is obtained from the linear manifold D^ν of infinitely differentiable functions finite in Ω by completion with respect to the scalar product

$$(u_1, u_2)^\nu = \int_\Omega \sum_{\substack{\alpha_1+\dots+\alpha_n=\nu \\ \beta_1+\dots+\beta_n=\nu}} b^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_n} \frac{\partial^\nu u_1}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \frac{\partial^\nu u_2}{\partial x_1^{\beta_1}\cdots\partial x_n^{\beta_n}}\, d\Omega. \tag{6}$$

In H^ν consider the operator Π^ν, defined by the formula

$$\Pi^\nu u = -L_\nu^{-1} M_\nu u, \qquad u \in D^\nu,$$

where

$$M_\nu = -\sum_{\substack{\alpha_1+\dots+\alpha_n=\nu\\ \beta_1+\dots+\beta_n=\nu}} \frac{\partial^\nu}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\left(C^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_n}\frac{\partial^\nu}{\partial x_1^{\beta_1}\cdots\partial x_n^{\beta_n}}\right);$$

$$C^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_n} = \sum_{\gamma_1+\dots+\gamma_n=\nu} a^{\alpha_1,\dots,\alpha_n}_{\gamma_1,\dots,\gamma_n} b^{\gamma_1,\dots,\gamma_n}_{\beta_1,\dots,\beta_n}, \tag{7}$$

and L_ν^{-1} is the operator inverse to the operator L_ν under the boundary conditions (4).

Theorem 1. *The operator Π_ν on the linear manifold D_ν is symmetric and bounded in the scalar product (1).*

Theorem 2. *The operator Π^ν on the linear manifold D^ν is symmetric and bounded in the scalar product (6).*

The continuous extensions to the whole space of the operator Π_ν in H_ν, and of the operator Π^ν in H^ν, will be denoted by the same letters.

Let now λ_i and $\mathbf{v}^{(i)}$ $(i = 1, 2, \dots, N)$ be the system of eigenvalues and eigenvectors of the matrix A. By D_ν^A we denote the linear manifold of vectors $\mathbf{v}(x)$ of the form

$$\mathbf{v}(x) = \sum_{k=1}^{N} \alpha_k(x)\mathbf{v}^{(k)},$$

where $\alpha_k(x)$ $(k = 1, 2, \dots, N)$ are smooth functions satisfying the condition

$$\mathbf{v}^{(k)} \operatorname{grad}_\nu \alpha_k(x) = 0 \qquad (k = 1, 2, \dots, N);$$

$$\operatorname{grad}_\nu \alpha_k(x) \equiv \left\{\frac{\partial^\nu \alpha_k(x)}{\partial x_1^{\beta_1}\cdots\partial x_n^{\beta_n}}\right\} \qquad (\beta_1 + \dots + \beta_n = \nu).$$

The closure of D_ν^A in the metric of H_ν will be denoted by H_ν^A. It can be shown that the space H_ν^A obtained in this way is an invariant subspace for the operator Π_ν in H_ν, and that on this subspace Π_ν

coincides with the operator matrix A. Put

$$H_\nu = H_\nu^A \oplus H_\nu^*.$$

By $H_\nu^{(1)}$ and $H^\nu_{(1)}$ we denote the subspaces that correspond to the discrete parts of the spectra respectively for the operator Π_ν in H_ν^* and for the operator Π^ν in H^ν. Further, put

$$H_\nu^{**} = H_\nu^* \ominus H_\nu^{(1)}, \qquad H_*^\nu = H^\nu \ominus H^\nu_{(1)}.$$

Theorem 3 (on the spectral equivalence of two operators).
1°. The sets of spectral points of the operator Π_ν in H_ν^* and of the operator Π^ν in H^ν coincide:

$$S(\Pi_\nu) = S(\Pi^\nu).$$

2°. The sets of points of the limiting spectrum of the operator Π_ν in H_ν^* and of the operator Π^ν in H^ν coincide.

3°. The eigenvalues of the operator Π_ν in H_ν^* and of the operator Π^ν in H^ν coincide, together with their multiplicities.

4°. In the case of a continuous spectrum, from a complete system of proper differentials of the operator Π_ν in H_ν^{**} one can construct a complete system of proper differentials for the operator Π^ν in H_*^ν, and conversely.

In the case when Ω is a finite domain bounded by the surface of the unit sphere with center at the origin, the following holds.

Theorem 4. The operator Π^ν in H^ν has a complete system of polynomial eigenfunctions, i.e. the spectrum of the operator Π^ν in H^ν is purely point.

We now prove the theorem on membership of points in the spectrum. For this purpose we introduce the following notation. By E^+, $\{E^-\}$ we denote the set of those values of the numerical parameter λ for which the differential operator

$$\mathcal{L}_\nu^{(+\lambda)}[u] \equiv Mu + \lambda Lu, \qquad \{\mathcal{L}_\nu^{(-\lambda)}[u] \equiv M_\nu u - \lambda L_\nu u\}$$

with constant coefficients is not elliptic in Ω. By E we denote the set of all λ^2 for which λ belongs to the set-theoretic sum $E^+ + E^-$. Put $L_{2\nu} = L_\nu^2$, $M_{2\nu} = M_\nu^2$. Then the following is true.

Theorem 5. If $\lambda_0^2 \in E$, then λ_0^2 is a spectral point for the operator $\Pi^{2\nu}$ in $H^{2\nu}$.

Proof. Indeed, let $\lambda_0^2 \in E$; then λ_0 belongs to one of the sets E^+, E^-. For definiteness suppose $\lambda_0 \in E^-$. Then the operator $\mathcal{L}_\nu^{(-\lambda_0)}$ is not elliptic and, consequently, is weaker than the elliptic operator L_ν, since, according to a theorem of Hörmander ([8]), a differential operator P is elliptic if and only if it is stronger than any operator whose order is not higher than the order of P. Thus, there exists a sequence of infinitely differentiable finite functions in Ω, $u_n(x) \in C_0^\infty(\Omega)$, such that

$$\lim_{n\to\infty} \|\mathcal{L}_\nu^{(-\lambda_0)} u_n\|_{L_2} = 0, \qquad \lim_{n\to\infty} \|L_\nu u_n\|_{L_2} \neq 0,$$

and since

$$\{\|u_n\|^{2\nu}\}^2 = (L_{2\nu}u_n, u_n)_{L_2} = (L_\nu u_n, L_\nu u_n)_{L_2} = \|L_\nu u_n\|^2_{L_2},$$

it follows that

$$\lim_{n\to\infty} \|u_n\|^{2\nu} \neq 0.$$

On the other hand, putting $(\Pi^{2\nu} - \lambda_0^2 E)u_n = v_n$, we shall have

$$\{\|(\Pi^{2\nu} - \lambda_0^2 E)u_n\|^{2\nu}\}^2 = ([M_{2\nu} - \lambda_0^2 L_{2\nu}]u_n, v_n)_{L_2} =$$

$$= ([M_\nu + \lambda_0 L_\nu][M_\nu - \lambda_0 L_\nu]u_n, v_n)_{L_2} =$$

$$= (\mathcal{L}_\nu^{(-\lambda_0)}u_n, \mathcal{L}_\nu^{(+\lambda_0)}v_n) \leq \{\|\mathcal{L}_\nu^{(-\lambda_0)}u_n\|_{L_2}\}^{1/2}\{\|\mathcal{L}_\nu^{(+\lambda_0)}v_n\|_{L_2}\}^{1/2} \leq$$

$$\leq c\{\|\mathcal{L}_\nu^{(-\lambda_0)}u_n\|_{L_2}\}^{1/2}\{\|L_\nu v_n\|_{L_2}\}^{1/2} = c\{\|\mathcal{L}_\nu^{(-\lambda_0)}u_n\|_{L_2}\}^{1/2}\{\|v_n\|^{2\nu}\}^{1/2}.$$

or, in final form,

$$\|v_n\|^{2\nu} \leqslant C\,\|\mathcal{L}_\nu^{(-\lambda_0)}u_n\|_{L_2}.$$

Thus we obtain $\lim_{n\to\infty} \|(\Pi^{2\nu} - \lambda_0^2 E)u_n\|^{2\nu} = 0$, whereas $\lim_{n\to\infty} \|u_n\|^{2\nu} \neq 0$, whence it follows that $\lambda_0^2 \in S(\Pi^{2\nu})$. The theorem is proved.

Remark. From the theorem just proved, in particular, it follows that the spectrum of the operator $B^2 = \Delta^{-2}\dfrac{d^4}{dt^4}$ in $H_B^*(\Omega)$ (see (8)) coincides with the segment $[0, 1]$ for any domain Ω.

2. Let us now consider the following nonhomogeneous boundary-value problem:

$$M_\nu u - \lambda^* L_\nu u = f(x_1, \dots, x_n), \tag{8}$$

$$u\big|_\Gamma = \frac{\partial u}{\partial n}\bigg|_\Gamma = \dots = \frac{\partial^{\nu-1}u}{\partial n^{\nu-1}}\bigg|_\Gamma = 0. \tag{9}$$

Let $f \in H^\nu$ and suppose that $M_\nu F = f,\ F \in H^\nu$.

Assume also that λ^* does not coincide with any eigenvalue of the operator Π^ν in H^ν. Then the following theorem is true, which shows the connection of the boundary-value problem (8), (9) with the nature of the spectrum of the operator Π^ν in H^ν.

Theorem 6. *If the spectrum of the operator Π^ν in H^ν is purely point spectrum and if the numerical series converges*

$$\sum_{k=1}^{\infty} \frac{F_k^2}{(1-\lambda^*/\lambda_k)^2}, \tag{10}$$

where F_k are the coefficients of the expansion of the function F in a series with respect to the eigenfunctions of the operator Π^ν in H^ν, then the boundary-value problem (8), (9) has a unique solution in H^ν.

Computing Center
of the Academy of Sciences of the ArmSSR

Yerevan State University

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Note: Figure translations are in progress. See original paper for figures.