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Ya. B. Rutickii

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Abstract

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Ya. B. Rutickii

SCALES OF ORLICH SPACES AND INTERPOLATION THEOREMS

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In recent years a number of works have appeared devoted to various generalizations and analogues of the interpolation theorems of M. Riesz. Some of these works are based on the study of the notion, introduced by S. G. Krein [1], of a scale of Banach spaces. In this connection, various scales of Banach spaces are of interest. In the present paper one of the methods for constructing scales of Orlicz spaces is indicated.

The definitions and notation from [2] are used in the paper.

1. Lemma. Let $0 < \tau < 1$. For any positive a_1, a_2, b_1, b_2 the inequality

$$(a_1 + b_1)^{1-\tau}(a_2 + b_2)^{\tau} \geq a_1^{1-\tau}a_2^{\tau} + b_1^{1-\tau}b_2^{\tau}. \tag{1}$$

Inequality (1) makes it possible to prove the following assertions.

Theorem 1. Let $N_0(v)$ and $N_1(v)$ be given N-functions. The function $N_\tau(v)$ $(0 \leq \tau \leq 1)$, whose inverse is defined by the equality

$$N_\tau^{-1}(u) = [N_0^{-1}(u)]^{1-\tau}[N_1^{-1}(u)]^{\tau} \quad (u \geq 0), \tag{2}$$

is an N-function.

Theorem 2. Let $M(u)$ be a given N-function. The function $\widetilde{M}_\tau(u)$ $(0 < \tau \leq 1)$, whose inverse is defined by the equality

$$\widetilde{M}_\tau^{-1}(v) = v^{1-\tau}[M^{-1}(v)]^{\tau} \quad (v \geq 0), \tag{3}$$

is an N-function.

The N-functions $M_\tau(u)$ and $\widetilde{N}_\tau(v)$, complementary to the N-functions $N_\tau(v)$ and $\widetilde{M}_\tau(u)$ defined by equalities (2) and (3), can be found explicitly only in rare cases. However, the following assertions hold.

Theorem 3. Let $M_0(u), M_1(u)$, and $M_\tau(u)$ be N-functions complementary to the N-functions $N_0(v), N_1(v)$, and $N_\tau(v)$ from Theorem 1. The N-function $\widetilde{M}_\tau(u)$, whose inverse is defined by the equality

$$\widetilde{M}_\tau^{-1}(v) = [M_0^{-1}(v)]^{1-\tau}[M_1^{-1}(v)]^{\tau} \quad (0 \leq \tau \leq 1,\ v \geq 0), \tag{4}$$

is equivalent to the N-function $M_\tau(u)$.

Theorem 4. Let $N(v)$ and $\widetilde{N}_\tau(v)$ be N-functions complementary to the N-functions $M(u)$ and $\widetilde{M}_\tau(u)$ from Theorem 2. The N-function

$$N_\tau(v) = N\left[\,|v|^{1/\tau}\,\right] \quad (0 < \tau \leq 1) \tag{5}$$

is equivalent to the N-function $\widetilde{N}_\tau(v)$.

We shall say that an N-function $M_0(u)$ is "strictly smaller" than an N-function $M_1(u)$ if

$$\lim_{u\to\infty} \frac{M_0(ku)}{M_1(u)} < \infty \tag{6}$$

for every $k > 0$. An N-function $M_0(u)$ is "essentially smaller"than an N-function $M_1(u)$ if

$$\lim_{u\to\infty} \frac{M_0(ku)}{M_1(u)} = 0$$

for every $k > 0$.

Theorem 5. If the N-function $M_0(u)$ is "strictly smaller" ("essentially smaller") than the N-function $M_1(u)$, then the analogous relation, for $0 \leq \tau_1 < \tau_2 \leq 1$, also holds for the N-functions $\widetilde{M}_{\tau_1}(u)$ and $\widetilde{M}_{\tau_2}(u)$ defined in Theorem 3.

Theorem 6. Let $\widetilde{M}_\tau(u)$ be the N-function defined in Theorem 2. Then, for $0 < \tau_1 < \tau_2 \leq 1$, the N-function $\widetilde{M}_{\tau_1}(u)$ is "essentially smaller" than the N-function $\widetilde{M}_{\tau_2}(u)$.

If the N-functions $M_0(u)$ and $M_1(u)$ in Theorem 3 ($M(u)$, in the case of Theorem 2) satisfy the Δ_2-condition or the Δ'-condition, then all the "intermediate" N-functions $\widetilde{M}_\tau(u)$ defined by equalities (3) and (4) satisfy the corresponding condition as well.

2. Let $M_0(u)$ and $M_1(u)$ be two given N-functions, and let $N_0(v)$ and $N_1(v)$ be their complements. We shall assume that one of the functions, for instance $M_0(u)$, is "strictly smaller" than the N-function $M_1(u)$. Let $M_\tau(u)$ be the N-functions complementary to the N-functions $N_\tau(v)$ defined in Theorem 1. Finally, let G be a bounded closed set in n-dimensional Euclidean space. We shall consider the spaces E_{M_τ}—the closures, in the Orlicz space $L^*_{M_\tau}(G)$, of the set of bounded functions.

It follows from Theorem 5 that, for $0 \leq \tau_1 < \tau_2 \leq 1$,

$$E_{M_{\tau_2}} \subset E_{M_{\tau_1}}.$$

Thus the spaces E_{M_τ} $(0 \leq \tau \leq 1)$ form a family of mutually embedded spaces. Since in each of them the set of functions bounded on G is dense, $E_{M_{\tau_2}}$ is dense in $E_{M_{\tau_1}}$ if $\tau_2 > \tau_1$.

The family of spaces E_{M_τ} $(0 \le \tau \le 1)$ will be called a **scale of Orlicz spaces** connecting the spaces E_{M_0} and E_{M_1}. If both functions $M_0(u)$ and $M_1(u)$ satisfy the Δ_2-condition, then, in view of the remark made above, one may speak of a scale of spaces $L_{M_\tau}(G)$, densely embedded in one another and connecting the spaces $L_{M_0}(G)$ and $L_{M_1}(G)$.

Theorem 7. Let $0 \le \tau_1 < \tau < \tau_2 \le 1$. For every function $x(s)$ from $E_{M_{\tau_2}}$ the inequality

$$\|x\|_{M_\tau} \le \|x\|_{M_{\tau_1}}^{\frac{\tau_2-\tau}{\tau_2-\tau_1}} \|x\|_{M_{\tau_2}}^{\frac{\tau-\tau_1}{\tau_2-\tau_1}}. \tag{7}$$

holds.

Theorem 7 asserts the logarithmic convexity of the norm of elements of E_{M_τ} as a function of the parameter τ.

If $M_0(u) \le M_1(u)$ for all $u \ge 0$ (this can always be achieved, if $M_0(u)$ is “strictly smaller” than $M_1(u)$, by passing, if necessary, to equivalent N-functions), then the inequality

$$\|x\|_{\tau_1} \le \|x\|_{\tau_2} \quad (\tau_1 < \tau_2,\ x(s) \in E_{M_{\tau_2}})$$

holds.

In this case the scale E_{M_τ} is a **normal** scale in the sense of S. G. Krein’ s (^{3}) scale.

The family of functions $N_\tau(v)$ defined in Theorem 4 makes it possible to construct a scale of Orlicz spaces E_{M_τ} $(0 < \tau \le 1)$ connecting the space $L_1(G)$ of summable functions on G with the given Orlicz space E_M. This means that one can specify such a family of spaces E_τ $(0 \le \tau \le 1)$ that $E_0 = L_1$, $E_\tau = E_{M_\tau}$ $(0 < \tau < 1)$, and $E_1 = E_M$.

Analogously, the family of functions $N_\tau(v)$ whose inverses are defined by the formula

$$N_\tau^{-1}(u) = u^{1-\tau}[N^{-1}(u)]^\tau \quad (u \ge 0),$$

allows one to construct a scale of Orlicz spaces E_{M_τ} $(0 < \tau < 1)$, connecting the space $L_\infty(G)$ of essentially bounded functions on G with the given Orlicz space E_M. And for such scales the theorem on logarithmic convexity of the norm is valid.

3. Let G be a bounded closed set in n-dimensional Euclidean space, and let G_1 be such a set in m-dimensional space. We shall consider operators A, defined on functions $x(s)$ $(s \in G)$ with values in the set of functions $\varphi(t)$ $(t \in G_1)$. Let $E_\tau(G)$ $(0 \le \tau \le 1)$, $F_\tau(G_1)$ $(0 \le \tau \le 1)$ be two scales of

Orlicz spaces described above. The spaces E_0, E_1, F_0, or F_1 may be the spaces L_1 or L_∞. Then the following assertion holds.

Theorem 8. *Let a homogeneous and additive operator A act from E_0 into F_0, and from E_1 into F_1, and suppose that*

$$\|Ax\|_{F_0} \le K_0 \|x\|_{E_0}, \qquad \|Ax\|_{F_1} \le K_1 \|x\|_{E_1}.$$

Then for any τ, $0 < \tau < 1$, the operator A acts from E_τ into F_τ and

$$\|Ax\|_{F_\tau} \le 8 K_0^{1-\tau} K_1^{\tau} \|x\|_{E_\tau}.$$

The proof of this theorem does not use the general theory of analytic scales, developed by S. G. Krein, and is based on passage to complex Orlicz spaces and on the use of the well-known idea of applying Hadamard' s three-lines theorem. The question of the possibility of applying the theory of analytic scales in the case of Orlicz spaces remains open. Apparently, the scales E_{M_τ} can be included in this theory in the same way as was done in (^{1}) for the scales of spaces L_p.

For scales of Orlicz spaces, theorems are proved that are analogous to the theorem of M. A. Krasnosel' skii (^{4}) on the interpolation of completely continuous operators in L_p. We formulate one of these theorems.

Theorem 9. *Suppose that under the assumptions of Theorem 8 the operator A is completely continuous as an operator from E_1 into F_1, if $E_0 \subset E_\tau \subset E_1$, or completely continuous as an operator from E_0 into F_0, if $E_1 \subset E_\tau \subset E_0$. Suppose that F_1 in the first case, or F_0 in the second, is not the space L_∞. Then for every τ, $0 < \tau < 1$, the operator A is completely continuous as an operator from E_τ into F_τ.*

From Theorems 8 and 9 there follow new theorems on conditions for continuity and complete continuity of integral operators of the type of the theorems of L. V. Kantorovich (^{5}). For Orlicz spaces, some theorems of this kind without the use of interpolation theorems were obtained in (^{6}, ^{7}).

The interpolation theorems may also find applications in establishing new embedding theorems and in other questions (for example, in the study of fractional powers of operators).

The notion of a scale and the interpolation theorems are also proved for coordinate Orlicz spaces l_M.

4. The notion "strictly less," introduced for N-functions by relation (6), turns out to be natural in many questions. As an example we give the following assertions.

Theorem 10. *In order that from the relation*

$$\lim_{n\to\infty} \int_G M_1[x_n(s)]\,ds = 0$$

there follow the relation

$$\lim_{n\to\infty} \|x_n\|_{M_0} = 0,$$

it is necessary and sufficient that the N*-function* $M_0(u)$ *be "strictly less" than the* N*-function* $M_1(u)$*.*

Theorem 11. In order that, for every $a \geqslant 1$, the inequality

$$\|x\|_{M_1} \leqslant a$$

imply the inequality

$$\int_G M_0[x(s)]\,ds \leqslant b(a),$$

it is necessary and sufficient that the N-function $M_0(u)$ be "strictly smaller" than the N-function $M_1(u)$.

A consequence of these theorems are the known assertions that convergence in norm and in mean, and also boundedness in norm and in mean in the Orlicz space L_M^*, are equivalent if and only if the N-function $M(u)$ satisfies the Δ_2-condition.

Voronezh Civil Engineering Institute

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Note: Figure translations are in progress. See original paper for figures.

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