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Abstract

Full Text

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ON A CONSTRUCTIVE CHARACTERISTIC OF CERTAIN CLASSES OF CONTINUOUS FUNCTIONS DEFINED ON A SEPARABLE METRIC SPACE

(Presented by Academician S. N. Bernstein on 27 XI 1962)

Let Q be an arbitrary separable metric space with metric $\rho(x, y)$, let $\omega(t)$ be any modulus of continuity, defined for $0 \le t \le d(Q)$, where $d(Q)$ is the diameter of the space Q (see ([1]), p. 109), and let $H_\omega(Q)$ be the class of all real-valued functions $f(x)$ bounded on Q for which

$$|f(x) - f(y)| \le \omega\{\rho(x, y)\}. \tag{1}$$

In the case where the space Q is the real number axis with the usual metric $\rho(x, y) = |x - y|$, the natural apparatus for approximating the functions considered in approximation theory is furnished by entire functions $g_\sigma(x)$ of finite degree $\sigma \ge 0$, bounded on the whole axis Q. One of the most essential properties of such functions is the well-known inequality of S. N. Bernstein

$$|g'_\sigma(x)| \le \sigma \sup_{-\infty<x<\infty} |g_\sigma(x)|,$$

which, under the normalization $\sup_{-\infty<x<\infty} |g_\sigma(x)| = 1$, is written in the following equivalent form:

$$|g_\sigma(x) - g_\sigma(y)| \le \sigma\rho(x, y). \tag{2}$$

In this form this inequality may be taken as the definition of the class $G'_\sigma(Q)$ of all real-valued functions $g_\sigma(x)$, considered already in an arbitrary metric space Q with some metric $\rho(x, y)$, and, like entire functions of finite degree, playing the role of a constructive element in the study of uniform approximations of arbitrary bounded and uniformly continuous functions $f(x)$ in the general case.

For the best uniform approximations

$$A'_\sigma(f; Q) = \inf_{g_\sigma(x) \in G'_\sigma(Q)} \sup_{x \in Q} |f(x) - g_\sigma(x)| \tag{3}$$

of functions $f(x) \in H_\omega(Q)$ the following theorem is valid.

Theorem 1. *Whatever the modulus of continuity $\omega(t)$ and the separable metric space Q, in order that $f(x) \in H_{c\omega}(Q)$ for some $c > 0$, it is necessary and sufficient that, for $0 \le \sigma < \infty$,*

$$A'_\sigma(f;Q) = O\left\{\max_{0\le t\le d(Q)}[\omega(t) - \sigma t]\right\}. \tag{4}$$

For the proof of sufficiency in this theorem it should be noted that, in the general case, the inequality

$$|f(x) - f(y)| \le 2\rho(x,y)\,\delta\{\rho(x,y)\},$$

where $\delta(t)$ is the root of the equation

$$\frac{2A'_\sigma(f;Q)}{\sigma} = t.$$

The proof of necessity is based on the following general assertion.

Theorem 2. *Whatever the separable metric space Q with metric $\rho(x,y)$, for every real function $f(x)$ bounded on Q and continuous with respect to the given metric, the equality*

$$A'_1(f;Q) = \frac{1}{2}\sup_{x,y\in Q}\{|f(x) - f(y)| - \rho(x,y)\}. \tag{5}$$

Let us note a consequence of Theorem 1 for one known class of functions, which, even in the case of real functions of one real variable within the framework of the classical theory of approximation by polynomials (or by entire functions of finite degree), does not have an exhaustive constructive characterization.

Theorem 3. *Let Q be an arbitrary separable metric space with metric $\rho(x,y)$. In order that a real function $f(x)$ bounded on Q satisfy the condition*

$$f(x) - f(y) = O\{\rho(x,y)|\ln\rho(xy)|\}, \qquad x,y\in Q, \tag{6}$$

it is necessary and sufficient that, for some positive $q < 1$ and $0 \le \sigma < \infty$, the relation

$$A'_\sigma(f;Q) = O(q^\sigma) \tag{7}$$

hold.

Let us note that in the case when the modulus of continuity $\omega(t)$ is convex, Theorem 1 admits the following refinement:

Theorem 4. *Whatever the convex (upward) modulus of continuity $\omega(t)$ and the separable metric space Q, in order that $f(x) \in H_\omega(Q)$, it is necessary and sufficient that, for every $\sigma > 0$, the inequality*

$$A'_\sigma(f;Q) \leq \frac{1}{2} \max_{0 \leq t \leq d(Q)} \{\omega(t) - \sigma t\} \tag{8}$$

hold.

The necessity in Theorem 4, which also holds without the additional assumption of convexity of $\omega(t)$, follows from Theorem 2.

The sufficiency can be obtained with the aid of the following simple lemma.

Lemma. *If $\omega_1(t)$ and $\omega_2(t)$ are arbitrary convex moduli of continuity for which, for every $M \geq 0$, the difference $\omega_1(t) - M\omega_2(t)$ is either monotone, or has an increment that changes sign once from plus to minus as t increases, then, whatever h on the interval $[0,d]$, the relation*

$$\inf_M \left\{ \max_{0 \leq t \leq d} [\omega_1(t) - M\omega_2(t)] + M\omega_2(h) \right\} = \omega_1(h) \tag{9}$$

holds.

The conditions of this lemma, in particular, are always fulfilled when $\omega_2(t) = t$. There exist examples (see ([2])) showing that even for $\omega_2(t) = t$, without the assumption of convexity of the function $\omega_1(t)$, relation (9) may fail to hold. From the property of any modulus of continuity $\omega(\lambda t) \leq (\lambda + 1)\omega(t)$ it follows directly that, in the general case, the inequality

$$\inf_M \left\{ \max_{0 \leq t \leq d} [\omega_1(t) - M \cdot t] + Mh \right\} \leq 2\omega_1(h) \tag{10}$$

is always valid.

In conclusion, we note that in the case when the space Q is the real numerical half-axis with the usual metric $\rho(x,y) = |x - y|$, for the best approximations $A'_\sigma(f;Q)$ in the class of functions $f(x)$ convex and monotone on the half-axis Q, a duality principle holds, established by the author in ([3]).

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[1] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Moscow, 1960. [2] A. F. Timan, DAN, 140, No. 2, 307 (1961). [3] A. F. Timan, DAN, 149, No. 5 (1963).