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Abstract

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THE METHOD OF STEEPEST DESCENT IN A HILBERT MODULE OVER A FINITE-DIMENSIONAL COMPLEX K-SPACE

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1. The set V of all m-dimensional vectors $v = \{\lambda_n\} = \sum_{n=1}^{m} \lambda_n e_n$ with complex coordinates is naturally converted into a K^2-space (1), where by convergence in the K-space (2) Z of all m-dimensional vectors with real coordinates one means (o)-convergence, coinciding with coordinatewise convergence; the unit $1 = \{1\} = \sum_{n=1}^{m} e_n$. For $v = \{\lambda_n\}$, $w = \{\mu_n\}$ the product $vw = \{\lambda_n \mu_n\}$, and the quotient $v/w = \{\nu_n\}$, where $\nu_n = \lambda_n/\mu_n$ for $\mu_n \neq 0$ and $\nu_n = 0$ for $\mu_n = 0$. The functional J, defined by the equality

$$J\left(\sum_{n=1}^{m} \lambda_n e_n\right) = \sum_{n=1}^{m} \lambda_n,$$

is (o)-linear and strictly positive in Z.

2. Let X be an M_V-module (1,3). Since in X, by definition, products vx always exist for $|v| \leq \lambda 1$, and Z is an extended K-space of bounded elements, X is an absolute M_V-module (1,3). Let X simultaneously be a Banach space, and suppose that for all n the inequalities

$$\|xe_n\| \leq \|x\|. \tag{1}$$

hold. Define the **structural norm**

$$|x| = \sum_{n=1}^{m} \|xe_n\| e_n.$$

Then the inequalities

$$|x| \leq \|x\| 1, \qquad \|x\| \leq J|x|,$$

hold, whence follows the equivalence of boundedness of a set (or convergence of a direction (4)) in X in the sense of a Banach space and boundedness of a set (or, respectively, convergence of a direction) in the structural norm. From the coincidence of the meanings of convergence of directions follows the fact that X is an absolute module of type B_{K^2}(1,3). We shall call X a **Banach module** over V.

3. If X is an M_V-module and a Hilbert space (complex) with scalar product $[x, y]$, and if for all n the equalities

$$[x, ye_n] = [e_{nx}, ye_n],$$

hold, then, introducing the **structural product** (previously called in (1,3) "scalar")

$$(x, y) = \sum_{n=1}^{m} [x, ye_n]\, e_n,$$

we embed X in an absolute module of type H_{K^2} (1, 3) and call it a **Hilbert module** over V. The equalities hold

$$[x, y] = J(x, y), \qquad [e_{nx}, y] = [x, ye_n].$$

Since $|xe_n| \ll |x|$, $\|y\|^2 = J(|y|^2)$, the inequalities (1) hold, and the Hilbert module is a Banach module.

In what follows, depending on the meaning of the norm or product, we shall also denote X by one of the symbols B, B_V or H, H_V.

If x is disjoint from y in H_V, i.e. $|x|\,|y|$ in Z, then (3) $x \perp y$ in H_V, and, consequently, $x \perp y$ in H.

4. Let U be an operation taking $X = B_V$ into the Banach module Y over V. Introduce the kn-section of the operation U:

$$U_{kn}(x) = e_{kU}(e_{nx})$$

and the "projector with renumbering" in the space V:

$$\Pi_{kn}\left(\sum_i \lambda_i e_i\right) = \lambda_n e_k.$$

A linear transformation $A \sim (u_{kn})$ in the space V can be written as

$$A = \sum_{kn} u_{kn} \Pi_{kn}.$$

Let U be a linear (bounded) operation taking $X = B$ into the given Banach space Y. Then all sections U_{kn} are also linear (bounded) in the same sense,

$$u_{kn} = \|U_{kn}\| \leq \|U\|,$$

and the operation U is regular, taking $X = B_V$ into the Banach module Y, with (o)-linear majorant A:

$$U \ll A = \sum_{kn} u_{kn} \Pi_{kn}.$$

The estimate for U is preserved also in the case when Y is a Banach module over a finite-dimensional K^2-space of another dimension.

5. An operation U for which

$$U(vx) = vU(x)$$

for all x in the domain of definition of the given operation and all $v \in V$, will be called (v)-homogeneous. For its sections the equalities hold

$$U_{kn}(x) = \delta_{kn} e_{kU}(x) = \delta_{kn} U(e_{kx}), \qquad \delta_{kn} = 0 \ (k \neq n), \quad \delta_{nn} = 1.$$

In particular, for the identity operation I,

$$I_{kn} x = \delta_{kn} e_{kx}.$$

If U is invertible and (v)-homogeneous, then U^{-1} is also (v)-homogeneous. If the operation U is linear (bounded) in B and (v)-homogeneous, then a majorant for U is the operation of multiplication by an element of Z:

$$|U|(z) \leq u \cdot z, \qquad u = \sum_{n=1}^{m} u_{nn} e_n \in Z.$$

6. If $X = Y = H_V$, the operator U is self-adjoint in H, then $U^*_{kn} = U_{nk}$ in H, and the matrix (b_{kn}), where $b_{kn} = [U_{kn}x, x]$, is self-adjoint for all x. Construct the matrix (c_{kn}) by deleting from the matrix (b_{kn}) all rows and columns with those numbers l_i for which the projections $xe_{l_i} = 0$ and, consequently, $b_{l_i n} = b_{kl_i} = 0$ for all k, n. If U is positive definite in H, then (c_{kn}) corresponds to a positive definite quadratic form and $\Delta = \det(c_{kn}) \neq 0$, since Δ is the Gram determinant of the system of elements Ce_{nx} $(x \neq 0,\ n \neq l_i)$, where $C = \sqrt{U}$ in H; the system $\{Ce_{nx}\}$ $(n \neq l_i)$ is linearly independent by virtue of the invertibility of the operator C and the pairwise disjointness in H_V of the elements $e_{nx} \neq 0$ for $n \neq l_i$.

7. If $U = U^*$ in H and is (v)-homogeneous, then $U = U^*$ also in H_V, the matrix (b_{kn}) becomes diagonal,

$$b_{nn} = [Ux, xe_n], \qquad (Ux, x) = \sum_n b_{nn} e_n \in Z.$$

8. Let, in the equation $Ux = f$, the operator U be bounded and positive definite in H. We shall construct a minimizing sequence $\{x_p\}$ for the quadratic functional $F(x) = [Ux, x] - 2\operatorname{Re}[x, f]$ in the form

$$x_{p+1} = x_p - v_p y_p, \qquad y_p = Ux_p - f,$$

where $v_p = \{\varepsilon_{np}\}_n$. The expression $F(x_0 - vy_0) - F(x_0)$ is a function of the coordinates λ_j $(j \neq l_i)$ of the vector v:

$$\Phi = \sum_{kn} c_{kn} \lambda_n \bar{\lambda}_k - \sum_k d_k \bar{\lambda}_k - \sum_k \bar{d}_k \lambda_k,$$

where $k, n \neq l_i$, $d_k = [y_0, y_0 e_k]$, $c_{kn} = [Ue_{ny}0, y_0 e_k]$. The point of minimum is determined by the equations

$$\frac{\partial \Phi}{\partial \lambda_j} \equiv \sum_n b_{jn} \lambda_n - d_j = 0 \qquad (j, n \neq l_i),$$

the system is solvable. Thus, an algorithm of the method of steepest descent in the Hilbert module over V is constructed:

$$x_{p+1} = x_p - \sum_{n=1}^{m} \varepsilon_{np} e_{ny} p, \qquad y_p = Ux_p - f \qquad (p = 0, 1, \ldots); \tag{2}$$

the coefficients ε_{np} are found from the system of equations:

$$\varepsilon_{l_i p} = 0 \quad \text{when } y_p e_{l_i} = 0$$

(if such l_i exist),

$$\sum_n [Ue_{ny} p, y_{pe} k] \varepsilon_{np} = [y_p, y_{pe} k] \qquad (k, n \neq l_i). \tag{3}$$

Since any product of the form $\varepsilon y \equiv e1y$, where ε is a number, is one of the products of the form vy, where $v \in V$, it follows that $F(x_1) \leqslant F(\bar{x}_0 - \varepsilon y_0)$, and the first approximation x_1, obtained by formula (2), is energetically (^{5}) closer to the solution x^* of the equation $Ux = f$:

$$\|C(x_1 - x^*)\| \leqslant \|C(\bar{x}_1 - x^*)\| \qquad (C = \sqrt{U})$$

in comparison with the first approximation $\bar{x}_1 = x_0 - \varepsilon_0 y_0$ (where $[Uy_0, y_0]\varepsilon_0 = [y_0, y_0]$), which is obtained by the method of steepest descent (^{4}) in the Hilbert space H. Using this (cf. (^{4}), Ch. 15, § 1), we establish that $x_p \to x^*$,

$$\|C(x_p - x^*)\| \leqslant q^p \|C(x_0 - x^*)\|, \qquad \|x_p^n - x^*\| \leqslant q^p \|y_0\|/m, \tag{4}$$

where $q = (M - m)/(M + m)$; $M \geqslant m$ are the bounds of the operator U in H.

9. If, moreover, the operator U is (v)-homogeneous, then system (3) assumes the simplest form:

$$[Uy_p, y_{pe}n]\varepsilon_{np} = [y_p, y_{pe}n] \qquad (n \neq l_i),$$

the coefficients ε_{np} here are real. In this case the estimates (4) of convergence of the method can be strengthened:

$$\|e_n C(x_p - x^*)\| \leqslant q_n^p \|e_n C(x_0 - x^*)\|,$$

$$\|(x_p - x^*)e_n\| \leqslant q_n^p \|y_0 e_n\|/m_n \qquad (n = 1, \dots, m), \tag{5}$$

where $q_n = (M_n - m_n)/(M_n + m_n)$; $M_n \geqslant m_n$ are the bounds of the operator U in the subspace $H_n \subset H$ of all x that coincide with their projections xe_n. Obviously, $M \geqslant M_n \geqslant m_n \geqslant m,\ q_n \leqslant q$.

Below we shall establish conditions sufficient for carrying out a stationary process (7) of approximations, which is a modification of algorithm (2).

10. If the additive operation A maps some K-space Z into itself, is positive (in the sense of the structure) in it, and there also exists a posi-

a positive operation $(I - A)^{-1}$, defined on all of Z, then the series $\sum_{p=0}^{\infty} A^p z$ (o)-converges for every z.

11. If in some space X of type B_K with norms $x \in Z$ (in particular, in $X = B_V$) a regular operation T has an (o)-linear majorant $A \geq T$, and moreover $(I - A)^{-1} > 0$ in Z, then the algorithm $x_{p+1} = Tx_p + \psi$ (bk)-converges to the solution x^* of the equation $x = Tx + \psi$ with rate

$$x_p - x^* \leq A^p (I - A)^{-1} a \to 0, \tag{6}$$

where $a = x_0 - Tx_0 - \psi$.

12. In what follows $X = H_V$ and the operator $U = U^*$ in $X = H$. Hence follows the symmetry of the matrix (u_{kn}) (see above) and the self-adjointness of the operators U_{nn}. Denote $e_{kn} = e_k + (1 - \delta_{kn})e_n$. Compose the sets S_{kn} of all $x = xe_{kn}$ with norm $\|x\| = 1$. Introduce the notation:

    $$m'_{kn} = \inf_{S_{kn}}[Ux, x], \qquad m''_{kn} = \sup_{S_{kn}}[Ux, x], \qquad m_n = m'_{nn}, \qquad M_n = m''_{nn};$$

    $$q_{kn} = \max_{\nu=1,2} |m^{(\nu)}_{kn}| = \sup_{S_{kn}} |[Ux, x]|, \qquad q_n = q_{nn}.$$

    Then

    $$u_{kn} \leq q_{kn} = q_{nk}, \qquad u_{nn} = q_n \quad (u_{kn} = \|U_{kn}\|).$$

13. Introduce the operators

    $$(zU)x = \sum_{n=1}^{m} \varepsilon_n e_n Ux, \qquad T = I - zU, \qquad T_n = T_{nn},$$

    where all real numbers $\varepsilon_n \neq 0$. Then the equations $Ux = f$ and $x = Tx + \psi$ (where $\psi = zf$, $z = \{\varepsilon_n\}$) are equivalent to one another. The estimates

    $$\|T_n\| = \max_{\nu} |1 - \varepsilon_n m^{(\nu)}_{nn}|, \qquad \|T_{kn}\| = |\varepsilon_k| u_{kn} \leq |\varepsilon_k| q_{kn} \quad (k \neq n).$$

    are valid.

14. Suppose that for the operator U there is a possibility of such a choice of real coefficients $\varepsilon_n \neq 0$ and certain numbers $w_{kn} \geq u_{kn}$ for $k \neq n$ (for example, $w_{kn} = q_{kn}$) that the transformation $(I - A)^{-1} > 0$, where $A \sim (a_{kn})$,

    $$a_{nn} = \max_{\nu} |1 - \varepsilon_n m^{(\nu)}_{nn}|, \qquad a_{kn} = |\varepsilon_k| w_{kn} \quad (k \neq n),$$

    for example, $a_{kn} = \max_\nu |\delta_{kn} - \varepsilon_k m^{(\nu)}_{kn}|$. Then the algorithm

    $$x_{p+1} = x_p - \sum_{n=1}^{m} \varepsilon_n e_n (Ux_p - f) \tag{7}$$

    converges to x^* with rate (6), where

    $$a = \sum_{n=1}^{m} |\varepsilon_n| \|(Ux_0 - f)e_n\| e_n.$$

15. If the operator H is positive definite in H and (v)-homogeneous, then all $T_{kn} = 0$ for $k \neq n$, and therefore the choice of the indicated numbers ε_n becomes possible, for example, $\varepsilon_n = 2/(M_n + m_n)$. In this case $T(z) \leq Q \cdot z$, $Q = \{q_n\}$. We establish the estimates (5), valid also for the process (7) in the case under consideration.

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Note: Figure translations are in progress. See original paper for figures.