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Abstract

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MATHEMATICS

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INTEGER POINTS IN MULTIDIMENSIONAL ELLIPSOIDS

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Let r be a natural number ($r > 4$); let $Q(\mu_1, \mu_2, \dots, \mu_r)$ be a positive definite quadratic form with integer coefficients; let D be its determinant; let $M_1, M_2, \dots, M_r$ be natural numbers; let $b_1, b_2, \dots, b_r$ be integers; and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be arbitrary real numbers. Denote also by $\overline{Q}(\mu_1, \mu_2, \dots, \mu_r)$ the quadratic form conjugate to the form Q, and, if all the numbers $\alpha_1 M_1, \alpha_2 M_2, \dots, \alpha_{rM}r$ are rational, let H denote their least common denominator. Consider the following function of x ($x > 0$):

$$A(x) = \sum \exp\left[2\pi i \sum_{i=1}^{r} \alpha_i \mu_i\right],$$

where the summation is over all systems $(\mu_1, \mu_2, \dots, \mu_r)$ satisfying the conditions

$$Q(\mu_1, \mu_2, \dots, \mu_r) \le x; \tag{1}$$

$$\mu_i \equiv b_i \pmod{M_i} \quad (i = 1, 2, \dots, r). \tag{2}$$

(For the special case $\alpha_1 = \alpha_2 = \dots = \alpha_r = b_1 = b_2 = \dots = b_r = 0$, $M_1 = M_2 = \dots = M_r = 1$, this function is the number of integer points of the ellipsoid (1); in the general case we count the nodes of the "lattice" (2) with certain weights.) The principal term of this function corresponds to the "volume" of the ellipsoid (1). Define the remainder function by the relation

$$P(x) = A(x) - \frac{\pi^{r/2} \exp\left[2\pi i \sum_{i=1}^{r} \alpha_i b_i\right]}{\sqrt{D}\, M_1 M_2 \dots M_r\, \Gamma(r/2 + 1)} x^{r/2} \delta, \tag{3}$$

where $\delta = 1$ if all the numbers $\alpha_1 M_1, \dots, \alpha_{rM} r$ are integers, and $\delta = 0$ otherwise. This note is devoted to the question of estimating the order of growth of the function (3).

Using the methods of Landau (^{1}), Walvisch (^{2}), and Jarník (^{4,5}), one can reduce the problem of estimating (3) to the problem of estimating a sum of the form

$$\sum_{0<k\le\sqrt{x}} I_k,$$

where

$$I_k = x^{r/2-1} \frac{\log k + 2}{k^{r/2-1}} \tag{4}$$

in the general case, and

$$I_k = x^{r/4-1/2} \frac{\log k + 2}{R_k^{r/4}} \tag{5}$$

in the case when at least one of the numbers $\alpha_1, \alpha_2, \dots, \alpha_r$ is irrational; here R_k denotes the minimum of the expression

$$\overline{Q}\left(\frac{m_1}{M_1} - \alpha_1 k, \dots, \frac{m_r}{M_r} - \alpha_r k\right)$$

for all systems $(m_1, m_2, \dots, m_r)$ of integers. Using expression (4), we obtain the theorem:

Theorem 1. For all systems $(\alpha_1, \alpha_2, \dots, \alpha_r)$

$$P(x) = O\left(x^{r/2-1}\right). \tag{6}$$

In the case where all $\alpha_1, \alpha_2, \dots, \alpha_r$ are rational numbers, the following holds:

Theorem 2 (Walfish (3)). Suppose there exist relatively prime integers n, k $(k > 0,\ H \mid k)$, for which

$$S_{n,k} = \sum_{a_1, a_2, \dots, a_r = 1}^{k} \exp\left[2\pi i \frac{n}{k} Q(a_1 M_1 + b_1, \dots, a_{rM} r + b_r) + \right.$$

$$\left. + 2\pi i \sum_{i=1}^{r} \alpha_i (a_{iM} i + b_i)\right] \neq 0.$$

Then

$$P(x) = \Omega\left(x^{r/2-1}\right). \tag{7}$$

In the contrary case (i.e., if $S_{n,k} = 0$ for all admissible n, k)

$$P(x) = O\left(x^{r/4-1/10}\right).$$

Relation (7) can also be easily obtained from the following result, which is of independent interest (see (5)):

$$\int_0^x |P(y)|^2\,dy = \frac{\pi^r x^{r-1}}{4\pi^2 D \prod_{i=1}^r M_i^2 (r-1)\Gamma^2\left(\frac{r}{2}\right)} \sum_{\substack{k=1\\ H|k}}^{\infty} \sum_{\substack{(n,k)=1\\ n\neq 0}} \frac{|S_{n,k}|^2}{k^{2r-2}} + O(\varphi(x))$$

$$(\varphi(x) = x^{r-2} \text{ for } r > 5;\ \varphi(x) = x^3 \log^2 x \text{ for } r = 5).$$

Using expression (5) for small k and (4) for the remaining ones, we obtain the theorem:

Theorem 3. If at least one of the numbers $\alpha_1, \alpha_2, \dots, \alpha_r$ is irrational, then

$$P(x) = O\left(x^{r/2-1}\right). \tag{8}$$

This result cannot be improved without further restrictions on the numbers $\alpha_1, \alpha_2, \dots, \alpha_r$, as the following theorem shows:

Theorem 4. Let $\varphi(x)$ be a positive function tending monotonically to zero as x tends to infinity. Then there exists a system $(\alpha_1, \alpha_2, \dots, \alpha_r)$ for which (8) holds and

$$P(x) = \Omega\left(x^{r/2-1}\varphi(x)\right).$$

Taking into account the arithmetic properties of the systems $(\alpha_1, \alpha_2, \dots, \alpha_r)$ and using (4) and (5) more precisely, one can improve estimate (8).

Theorem 5. Let $\gamma = \gamma(\alpha_1, \alpha_2, \dots, \alpha_r)$ denote the upper bound of the set of all numbers β for which the inequality

$$\max_{i=1,2,\dots,r} (\alpha_{iM} ik) < \frac{1}{k^\beta}$$

is satisfied for infinitely many natural numbers k, $\gamma < \infty$, and

$$f = \left(\frac{r}{4} - \frac{1}{2}\right) \frac{2\gamma + 1}{\gamma + 1 - 2/r}.$$

Then

$$P(x) = O\left(x^{f+\varepsilon}\right)$$

for any arbitrarily small positive ε.

This estimate can be further improved by excluding a certain "small" number of systems $(\alpha_1, \alpha_2, \dots, \alpha_r)$.

Theorem 6. For almost all systems $(\alpha_1, \alpha_1, \dots, \alpha_r)$ (in the sense of Lebesgue measure in Euclidean space E_r) and for every $\varepsilon > 0$, one has

$$P(x) = O\left(x^{r/4} \log^{2r+1+\varepsilon} x\right).$$

The estimate of the main sum with terms of type (5), in view of the inequality

$$R_k \geq c \max_{i=1,2,\dots,r} (\alpha_i M_i k)^2 = cP_k^2$$

(the positive constant c does not depend on k), reduces to estimating the sum

$$x^{r/4-1/2} \log x \sum_{0<k\leq\sqrt{x}} \frac{1}{P_k^{r/2}}. \tag{9}$$

Let $\varphi(x)$ be a monotone positive function for which the series

$$\sum_{n=1}^{\infty} \frac{1}{\varphi(n)}$$

converges; let C and D be real numbers, $0 < C < D$, and let $W(C, D)$ be the set of all points of the space E_r for which $C \leq \alpha_i \leq D$ $(i = 1, 2, \dots, r)$. Further, for each set of integers $n \geq 0$, $m_1 > 0, \dots, m_r > 0$, denote by $\mathfrak{M}(n; m_1, \dots, m_r)$ the set of all $(\alpha_1, \alpha_2, \dots, \alpha_r) \in W(C, D)$ for which

$$|m_i\alpha_i - m_1\alpha_1| \leq \frac{2D}{2^n} \quad (i = 2, 3, \dots, r).$$

We shall say that the set $\mathfrak{M}(n; m_1, \dots, m_r)$ belongs to the class $[n; k_1, k_2, \dots, k_r]$ ($k_1, k_2, \dots, k_r$ are arbitrary nonnegative integers) if

$$2^{k_i} \leq m_i < 2^{k_i+1} \quad (i = 1, 2, \dots, r).$$

In the proof of Theorem 6, the following lemma is used in an essential way to estimate the sum (9).

Lemma. There exists a set $\mathfrak{N}(C, D) \subset W(C, D)$ of Lebesgue measure zero with the following property: for every $(\alpha_1, \alpha_2, \dots, \alpha_r) \in W(C, D) - \mathfrak{N}(C, D)$, there exists n_0 such that, for $n \geq n_0$, the system $(\alpha_1, \alpha_2, \dots, \alpha_r)$ belongs to no more than

$$2_1^k 2^{n(1-r)} \varphi(n+1) \prod_{i=1}^{r} \varphi(k_i + 1)$$

sets $\mathfrak{M}(n; m_1, \dots, m_r)$ of the class $[n; k_1, k_2, \dots, k_r]$.

Remark 1. Theorems 5 and 6 improve estimates for the abscissae of convergence of certain Dirichlet series (see (3)).

Remark 2. The results obtained can be generalized in the corresponding way to the case of rational $M_1, M_2, \dots, M_r, b_1, b_2, \dots, b_r$.

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Note: Figure translations are in progress. See original paper for figures.

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