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Abstract

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THE INCREASING PROPERTY OF DIRECT VALUES OF GENERALIZED POTENTIALS FOR THE COMPARISON FUNCTION AND THE PRINCIPAL FUNDAMENTAL SOLUTION OF A GENERAL SECOND-ORDER ELLIPTIC EQUATION

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Theorems on the increasing property of direct values of a potential play a large role in mathematical physics ([1]). However, up to now these theorems have been proved only for potentials of the Laplace operator ([2]). In the present work the increasing property is established for the direct values of the double-layer potential and the direct values of the conormal derivative of the single-layer potential for the comparison function and the principal fundamental solution of a general second-order elliptic equation. By the direct value of a double-layer potential is meant this same potential, considered as a function of the points of the surface on which its density is distributed. The direct values of the conormal derivative of a single-layer potential are defined analogously.

1°. Let, in the N-dimensional Euclidean space R_N, there be given an open domain g, whose boundary is a closed surface Γ. In the domain g we consider a general uniformly elliptic equation of second order

$$Mu = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x), \tag{1}$$

i.e., such that at any points $x = (x_1 \dots x_N) \in g$ one has

$$a_{ij} = a_{ji}, \qquad \sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq \alpha \sum_{i=1}^{N} \xi_i^2 \tag{2}$$

for arbitrary real $\xi_1, \dots, \xi_N$, where $\alpha > 0$ is a certain constant.

Suppose now that the surface Γ belongs to the class $A^{(n,\alpha)}$*,* $n > 2,\ 0 < \lambda \leq 1,$ *the coefficients of equation (1) belong to the class* $C^{(n-1,\lambda)}(g+\Gamma)$, and $c \leq 0$ in

$g + \Gamma$. Then, by virtue of a result of Gevrey (see ([3]), p. 77), the coefficients of equation (1) can be continued through Γ to the whole space R_N so that equation (1) is uniformly elliptic in the whole space, the coefficient $c(x)$ is nonpositive everywhere and strictly negative outside some bounded domain.

Definition. The function $G(x, y)$ is called the **principal fundamental solution** of equation (1), if it is a fundamental solution of equation (1)** in the whole space R_N and if there exist two constants $a > 0$ and $R > 0$ such that for $\gamma \geq R$

$$G(x, y) = O(e^{-a\gamma}), \qquad \frac{\partial G(x, y)}{\partial x_i} = O(e^{-a\gamma}), \qquad i = 1, \ldots, N,$$

where γ is the distance between the points x and y.

* For the definition of these classes, see ([3]), p. 10.
** For the definition of a fundamental solution, see ([3]), p. 26.

It is known that under the conditions formulated above the principal fundamental solution $G(x, y)$ of equation (1) exists ([4]).

Consider the double-layer potential for $G(x, y)$:

$$W(x) = \int_\Gamma \mu(y) P_{yG}(x, y)\, ds_y, \tag{3}$$

where $\mu(x)$ is the density of this potential,

$$P_{yG}(x, y) = \frac{\partial}{\partial \nu_y} G(x, y) = \frac{1}{a(y)} \sum_{i,j=1}^{N} a_{ij}(y) \frac{\partial G(x, y)}{\partial y_i} \cos(n, y_i)$$

is the conormal derivative of $G(x, y)$, while n is the exterior normal to Γ and

$$a(y) = \left[\sum_{i=1}^{N} \left(\sum_{j=1}^{N} a_{ij}(y) \cos(n, y_j) \right)^2 \right]^{1/2}.$$

The direct value W_{pr} on Γ of the double-layer potential $W(x)$ is the function of the points of the surface Γ

$$W_{\text{pr}} = W(x)\big|_{x \in \Gamma} = \int_\Gamma \mu(y) \left[P_{yG}(x, y) \right]_{x \in \Gamma} ds_y. \tag{4}$$

We have succeeded in proving that the direct value W_{pr} raises the smoothness of $\mu(x)$ by almost one unit, i.e. the following holds.

Theorem 1. If the density $\mu(x)$ belongs to the class $C^{(n-2,\lambda)}$ on Γ, then the direct value W_{pr} of the double-layer potential belongs to the class $C^{(n-1,\lambda')}$ on Γ, where $0 < \lambda' < \lambda$ is arbitrary; moreover, the norm W_{pr} in $C^{(n-1,\lambda')}(\Gamma)$ is estimated in terms of the norm of the density $\mu(x)$ in $C^{(n-2,\lambda)}(\Gamma)$:

$$\|W_{\text{pr}}\|_{C^{n-1,\lambda'}} = O(\|\mu\|_{C^{n-2,\lambda}}), \tag{5}$$

where the constant entering the O-term depends only on the norms of the coefficients of the equation $\|a_{ij}\|_{C^{n-1,\lambda}(g+\Gamma)}$, $\|b_i\|_{C^{n-1,\lambda}(g+\Gamma)}$, $\|c\|_{C^{n-1,\lambda}(g+\Gamma)}$, on the ellipticity constant α, on the choice of λ', and on the property of the surface Γ *, but does not depend on the density $\mu(x)$.

Remark 1. Theorem 1 remains valid if in (4) P takes the more general form:

$$P_{xu}(x) = \alpha(x)\frac{\partial u(x)}{\partial \nu_x} + \beta(x)u(x),$$

where $\alpha(x)$ and $\beta(x)$ are any two functions of the class $C^{(n-2,\lambda)}(\Gamma)$. Then the constant entering the O-term on the right-hand side of (5) also depends on the norms $\|\alpha\|_{C^{n-2,\lambda}(\Gamma)}$ and $\|\beta\|_{C^{n-2,\lambda}(\Gamma)}$.

2°. Suppose now that Γ belongs to the class $A^{(n,\lambda)}$, $0 < \lambda \leqslant 1$, $n \geqslant 2$, that a_{ij} belong to the class $C^{(n-1,\lambda)}$ in $g + \Gamma$, and that b_i and c belong to the class $C^{(n-2,\lambda)}$ in $g + \Gamma$, with $c \leqslant 0$. In this case one can also construct the principal fundamental solution $G(x, y)$ of equation (1), by virtue of the result of Gevrey noted above. For $G(x, y)$ consider the simple-layer potential:

$$V(x) = \int_\Gamma \nu(y)G(x, y)\, ds_y, \tag{6}$$

where $\nu(x)$ is the density of this potential.

* That is, it depends on the norm of the function defining the surface Γ in local coordinates.

The direct value V_{pr} on Γ of the conormal derivative of the simple-layer potential $V(x)$ is the following function of the points of the surface Γ:

$$V_{\text{pr}} = [P_{xV}(x)]_{x\in\Gamma} = \int_\Gamma v(y)\, [P_{xG}(x, y)]_{x\in\Gamma}\, ds_y. \tag{7}$$

We have proved that the direct value of the conormal derivative of the simple-layer potential increases the smoothness of the density $v(x)$ by almost one unit; that is, the following holds.

Theorem 2. If the density $v(x)$ belongs to the class $C^{(n-2,\lambda)}$ on Γ, then the direct value V_{pr} of the conormal derivative of the simple-layer potential belongs to the class $C^{(n-1,\lambda')}$ on Γ, where $0<\lambda'<\lambda$ is arbitrary, and the norm of V_{pr} in $C^{(n-1,\lambda')}(\Gamma)$ is estimated in terms of the norm of the density $v(x)$ in $C^{(n-2,\lambda)}(\Gamma)$:

$$\|V_{\text{pr}}\|_{C_{n-1,\lambda'}} = O(\|v\|_{C_{n-2,\lambda}}), \tag{8}$$

where the constant occurring in the term O depends only on the norms of the coefficients $\|a_{ij}\|_{C_{n-1,\lambda}(g+\Gamma)}$, $\|b_i\|_{C_{n-2,\lambda}(g+\Gamma)}$, $\|c\|_{C_{n-2,\lambda}(g+\Gamma)}$, the ellipticity constant a, the choice of λ', and the properties of the surface Γ, but does not depend on the density $v(x)$.

Remark 2. Theorem 2 remains valid if in (7) P has the more general form

$$P_{xu}(x) = \gamma(x)\frac{\partial u(x)}{\partial \nu_x} + s(x)u(x),$$

where $\gamma(x)$ and $s(x)$ are any two functions of the class $C^{(n-1,\lambda)}(\Gamma)$. Then the constant occurring in the term O on the right-hand side of (8) also depends on the norms $\|\gamma\|_{C_{n-1,\lambda}(\Gamma)}$ and $\|s\|_{C_{n-1,\lambda}(\Gamma)}$.

3°. Suppose now that a_{ij}, b_i, and c belong to the class $C^{(n,\lambda)}$ in R_N, $0<\lambda\le 1$, $0\le n$, and outside some bounded domain $c<-b^2$, where $b>0$. Then there also exists the principal fundamental solution $G(x,y)$ of equation (1). For $G(x,y)$ consider the volume potential

$$U(x) = \int_g Z(y)G(x,y)\,dy, \tag{9}$$

where g is any bounded domain, and $Z(x)$ is the density of this potential. It has been proved that the volume potential for the principal fundamental solution increases the smoothness of the density by almost two orders; that is, the following holds.

Theorem 3. Let the density $Z(x)$ belong to the class $C^{(n,\lambda)}(g+\Gamma)$ in $g+\Gamma$; then the volume potential $U(x)$ for the principal fundamental solution $G(x,y)$ belongs to the class $C^{(n+2,\lambda')}(g)$, where $0<\lambda'<\lambda$ is arbitrary, and the norm of $U(x)$ in $C^{(n+2,\lambda')}(g')$ is estimated in terms of the norm of $Z(x)$ in $C^{(n,\lambda)}(g+\Gamma)$:

$$\|U\|_{C_{n+2,\lambda'}(g')} = O(\|Z\|_{C_{n,\lambda}(g+\Gamma)}), \tag{10}$$

where g' is an arbitrary interior subdomain of the domain g, whose distance from the boundary Γ of the domain g is equal to d, and the constant occurring in the term O depends only on the norms of the coefficients $\|a_{ij}\|_{C_{n,\lambda}(g+\Gamma)}$, $\|b_i\|_{C_{n,\lambda}(g+\Gamma)}$, $\|c\|_{C_{n,\lambda}(g+\Gamma)}$, the ellipticity constant a, the choice of λ', and the number d, but does not depend on the density $Z(x)$.

Remark 3. Theorem 3 was proved by V. A. Il' in and I. A. Shishmarev for a smooth closed domain $g + \Gamma \in A^{(n+2,\lambda)}$ (5).

4°. It is known that to each equation (1) one can put in correspondence its own **comparison function**, defined by the formula

$$H(x,y) = \begin{cases} \dfrac{1}{(N-2)\omega_N \sqrt{A(y)}}\, \rho^{2-N}, & N \geqslant 3, \\ \dfrac{1}{2\pi\sqrt{A(y)}}\, \ln\dfrac{1}{\rho}, & N = 2, \end{cases}$$

where

$$\rho^2 = \sum_{i,j=1}^{N} A_{ij}(y)(x_i - y_i)(x_j - y_j),$$

and $A_{ij}(x)$ is the ratio of the algebraic complement of the element $a_{ij}(x)$ in the determinant $A(x) = |a_{ij}(x)|$ to the determinant $A(x)$ itself; ω_N is the area of the unit spherical surface of $N-1$ dimensions.

Let us now consider the potentials (3), (6), and (9), in which, instead of the principal fundamental solution $G(x,y)$, the comparison function $H(x,y)$ appears. Then Theorems 1, 2, and 3 are also valid; that is, more precisely, the following holds:

Theorem 4. *Suppose that all the conditions of Theorem 1 (or, respectively, of Theorem 2 or 3) are satisfied, except for the smoothness requirements on the coefficients $b_i(x)$ and $c(x)$. Under these conditions Theorem 1 (or, respectively, Theorems 2 and 3) is valid for the corresponding potential in which, instead of $G(x,y)$, the comparison function $H(x,y)$ stands; moreover, in this case the constants entering the estimates (5), (8), and (10) do not depend on the coefficients $b_i(x)$ and $c(x)$.*

Remark 4. The improving property of the direct values of potentials for the comparison function of the Laplace operator was proved by Kh. L. Smolitskii ([2]).

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Note: Figure translations are in progress. See original paper for figures.