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Abstract

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GEODESIC CORRESPONDENCE OF SEMI-REDUCIBLE RIEMANNIAN SPACES

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1. Two Riemannian spaces V_n and $\bar{V}_n$ are in geodesic correspondence if there exists a one-to-one mapping of one of them onto the other under which geodesics again pass into geodesics. If the space V_n has constant curvature K, then, according to the well-known theorem of Beltrami, the geodesically corresponding space $\bar{V}_n$ also has constant curvature $\bar{K}$. A geodesic mapping is regarded as trivial if it preserves not only geodesics but also the Riemannian connection. Investigating spaces V_n with positive-definite metric ds^2 (properly Riemannian V_n), Levi-Civita (^{1}) found the canonical form of the metric of a space V_n admitting nontrivial geodesic mappings. The problem of geodesic correspondence of properly Riemannian spaces was completely solved by A. S. Solodovnikov (^{2}). Among the Levi-Civita spaces he singled out a special class of spaces $V(K)$, including all spaces of constant curvature and most closely adjoining them in their geodesic properties. Let us note, in particular, that the spaces $V(K)$, like spaces of constant curvature, admit a continuous group of geodesic transformations (^{3}). As Solodovnikov established, the class of properly Riemannian spaces $V(K)$ is closed with respect to geodesic mappings, i.e. a properly Riemannian $V(K)$ can be in geodesic correspondence only with some space $V(\bar{K})$.

Both $V(K)$ and, in general, Levi-Civita spaces (with the exception of n-orthogonal ones) are special cases of semi-reducible Riemannian spaces (see (^{4})). Therefore, in passing to the study of geodesic mappings of spaces V_n with indefinite metric, it becomes necessary first of all to consider semi-reducible spaces. The present note is devoted to this problem. In it, in particular, it is established that the theorems on closedness with respect to geodesic mappings of the classes of spaces $V(K)$ and Levi-Civita spaces of the basic type (i.e. not $V(K)$) are valid independently of the condition $ds^2 > 0$. The only exceptions are certain spaces $V(0)$ possessing isotropic absolutely parallel vector fields.

If one assumes that corresponding points of the spaces V_n and $\bar{V}_n$ have the same coordinates, then the problem of geodesic correspondence reduces to finding spaces $\bar{V}_n$ having common geodesics with the given space V_n. A necessary and sufficient condition for this is the satisfaction of the equalities (^{5})

$$\bar{g}_{ij,k} = -(\bar{g}_{ij}\psi_{,k} + \tfrac{1}{2}\bar{g}_{ik}\psi_{,j} + \tfrac{1}{2}\bar{g}_{jk}\psi_{,i}), \tag{1}$$

where $\bar{g}_{ij}$ is the metric tensor of the space $\bar{V}_n$, the comma denotes the covariant derivative in V_n, and $\psi_{,i}$ is the gradient of some function ψ. Below, equations (1) are investigated in a semi-reducible space V_n under the assumption $\psi \neq$ const. The latter means that the correspondence between V_n and $\bar{V}_n$ is nontrivial.

2. A semi-reducible metric has the form (^{4})

$$ds^2 = ds_0^2(x^\alpha) + \sigma_1(x^\alpha)\, ds_1^2(x^{\alpha_1}) + \dots + \sigma_p(x^\alpha)\, ds_p^2(x^{\alpha_p}), \tag{2}$$

where the "principal part" ds_0^2 and the "additional" metrics $ds_1^2, \dots, ds_p^2$ each depend on their own variables, while the functions $\sigma_1, \dots, \sigma_p$ depend only on the coordinates from ds_0^2. It is assumed that in the domain under consideration $\sigma_\nu > 0$ and $\sigma_\nu/\sigma_\mu \neq$ const for $\nu \neq \mu$ ($\nu, \mu = 1, \dots, p$). The metric (2) is a Levi-Civita metric if

$$ds_0^2 = \sum_a e_a \prod_{b \neq a} |f_b - f_a| (dx^a)^2, \qquad \sigma_\nu = \prod_b |f_b^* - f_\nu|, \tag{3}$$

where $e_a = \pm 1$, each function f_b depends only on one coordinate x^b, and $f_\nu^* =$ const. A semi-reducible metric (2) defines a space $V(K)$ if, upon replacing in (2) all metrics ds_ν^2 ($\nu = 1, \dots, p$) by one-dimensional forms, one obtains the so-called associated metric ds^{*2}, having constant curvature K. In this case (2) is called a K-decomposition (^{3}).

Introducing the tensor $A_{ij} = e^\psi \bar{g}_{ij}$, we transform the system of equations (1) to the equivalent form

$$A_{ij,k} = -\frac{1}{2}(\psi_{,i}A_{jk} + \psi_{,j}A_{ik}). \tag{4}$$

As was shown by the author (^{4}), under certain algebraic conditions imposed on A_{ij}, system (4) determines semi-reducible V_n and is therefore called the system of equations of semi-reducibility. In this case, as a rule, A_{ij} is degenerate. Under the condition $\det(A_{ij}) \neq 0$, however, the solution of system (4) determines the metric $\overline{ds}^2 = e^{-\psi}A_{ij}dx^{idx^j}$, having geodesics in common with ds^2.

Lemma 1. *Every nondegenerate solution A_{ij} of system (4) in the space (2), satisfying the condition $A_{a\alpha_\nu} = 0$, has components*

$$A_{ab} = A_{ab}(x^a), \qquad A_{a\alpha_\nu} = 0, \qquad A_{\alpha_\nu\beta_\mu} = 0, \qquad A_{\alpha_\nu\beta_\nu} = C_\nu g_{\alpha_\nu\beta_\nu}.$$

Moreover $\psi = \psi(x^a)$ and

$$A_{ab,c} = -\frac{1}{2}(\psi_{,a}A_{bc} + \psi_{,b}A_{ac}), \qquad \det(A_{ab}) \neq 0,$$

$$u_{\nu}{}^{b}A_{ab} = C_{\nu}(u_{\nu,a} - \psi_{,a}), \qquad u_{\nu} = \ln\sigma_{\nu}, \qquad C_{\nu} = \text{const.}$$

Theorem 1. *If* ds^2 *has geodesics in common with the semi-reducible metric (2) and they satisfy the condition* $\bar{g}_{a\alpha_{\nu}} = 0$ $(\nu = 1, \dots, p)$, *then necessarily*

$$\overline{ds}^2 = \overline{ds}_0^2(x^a) + e^{-\psi(x^a)}(c_1\sigma_1 ds_1^2 + \dots + c_p\sigma_{pds}p^2), \tag{5}$$

where: a) $\overline{ds}_0^2$ *has geodesics in common with* ds_0^2*; b)* $u_{\nu}{}^{b}\bar{g}_{ab} = C_{\nu}e^{-\psi}(u_{\nu,a} - \psi_{,a})$, $u_{\nu} = \ln\sigma_{\nu}$. *Conversely, under conditions a) and b), the metrics (2) and (5) have geodesics in common.*

Thus, under the stated condition, $\overline{ds}^2$ is also semi-reducible with the same additional metrics ds_{ν}^2, and the question reduces to the geodesic correspondence of the principal parts ds_0^2 and $\overline{ds}_0^2$ under the additional condition b). We note that if, moreover, (2) is a K-decomposition, then (5) is some $\bar{K}$-decomposition. Indeed, along with (2) and (5), the associated metrics ds^{*2} and $\overline{ds}^{*2}$ will also possess common geodesics. If one of them has constant curvature K, then, by Beltrami' s theorem, the other has constant curvature $\bar{K}$.

3. Investigating in the space $V(K)$ the integrability conditions of system (4), we obtain the following lemma:

Lemma 2. *In the space* $V(K)$, *every nondegenerate solution* A_{ij} *of the equations of semi-reducibility (4) satisfies the condition*

$$\frac{1}{2}\psi_{,ij} + \frac{1}{4}\psi_{,i}\psi_{,j} = \bar{K}e^{-\psi}A_{ij} - Kg_{ij}, \qquad \bar{K} = \text{const.}$$

With the help of this condition it is possible to prove that the space $\overline{V}_n$ is $V(\bar{K})$, except only for the case when $K = \bar{K} = 0$, and the gradient $\psi_{,i}$

is isotropic both in V_n and in $\overline{V}_n$. In this exceptional case V_n is $V(0)$, admitting isotropic absolutely parallel vector fields. The space $\overline{V}_n$ also admits, and moreover only, isotropic absolutely parallel vector fields. Such spaces are denoted by $V(i)$. It is not difficult to find that the total number of linearly independent absolutely parallel vectors in $V(0)$ and in the corresponding space $V(i)$ is the same. As a result we obtain

Theorem 2. *Every space* $\overline{V}_n$ *having geodesics in common with a space* $V(K)$ *is also some space* $V(\overline{K})$, *or, in the exceptional case,* $V(i)$. *The metrics of the corresponding spaces are connected by the condition*

$$^1/_2\psi_{,ij} + {}^1/_4\psi_{,i}\psi_{,j} = \overline{K}\,\overline{g}_{ij} - Kg_{ij}.$$

The exceptional case is possible only for $K = \overline{K} = 0$ *and an isotropic gradient* $\psi_{,i}$*.*

It can be proved that any space $V(i)$ can have geodesics in common only with a space $V(0)$ or with another $V(i)$. Therefore, if all the spaces $V(i)$ are adjoined to $V(K)$, one obtains a class of Riemannian spaces with sign-indefinite metric that is closed with respect to geodesic mappings.

4. Let us next consider semi-reducible spaces V_n that are not spaces $V(K)$ or $V(i)$.

Lemma 3. *Every nondegenerate solution* A_{ij} *of system (4) in a semi-reducible space* V_n *that is not* $V(K)$ *or* $V(i)$ *satisfies the condition* $A_{\alpha_\nu} = 0$ $(\nu = 1, \dots, p)$.

Applying Theorem 1, we obtain the following assertion:

Theorem 3. *If a semi-reducible* V_n *with metric (2) is not a space* $V(K)$ *or* $V(i)$*, then every* $\overline{V}_n$ *having geodesics in common with* V_n *is also semi-reducible and has, in the same coordinate system, a metric (5).*

5. Since the Levi-Civita metric cannot determine spaces $V(i)$, applying the preceding results to it we obtain the theorem:

Theorem 4. *If* V_n *is a Levi-Civita space of the principal type (not* $V(K)$*), then every space* $\overline{V}_n$ *having geodesics in common with* V_n *is also a Levi-Civita space of the principal type. In some coordinate system the metric* ds^2 *has the form (2), (3), and* $d\overline{s}^2$ *has the form (5), where*

$$d\overline{s}_0^2 = e^{-\psi} \sum_a \frac{e_a}{|kf_a + l|} \prod_{b \neq a} |f_b - f_a|\,(dx^a)^2,$$

$$\psi = \ln \prod_a |kf_a + l|, \qquad k, l = \text{const}.$$

Thus, first, the class of Levi-Civita spaces of the principal type is closed with respect to geodesic mappings. Secondly, it follows from Theorem 4 that the matrix $(\overline{g}_{ij} - \lambda g_{ij})$ has simple elementary divisors and real roots. It is not difficult to prove that the latter condition is also sufficient for two metrics ds^2 and $d\overline{s}^2$, having common geodesics but different connections, to be Levi-Civita metrics.

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Note: Figure translations are in progress. See original paper for figures.