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By the product of two systems

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Abstract

Full Text

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THE PRODUCT OF BASES IN ARBITRARY DOMAINS

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By the product of two systems

$$p_n^{(s)}(z) = \sum_{k=0}^{\infty} p_{n,k}^{(s)} z^k \quad (s = 1, 2)$$

of functions analytic in some neighborhood of zero $|z| < \delta$, we shall mean the system

$$p_n(z) = \sum_{i=0}^{\infty} p_{n,i}^{(1)} p_i^{(2)}(z),$$

where uniform convergence of the series in some neighborhood of zero $|z| < \delta_1$ is assumed ([1,2]). The question of the product of bases in the spaces A_R of functions analytic in the disk $|z| < R$ has been considered in detail in papers ([2–4]).* Recently a number of papers ([5–7]) have appeared in which the product of bases in an arbitrary domain is studied, but for the special case of simple unit bases, i.e., bases of the form

$$p_n(z) = \sum_{k=0}^{n} p_{n,k} z^k, \qquad p_{n,n} = 1 \quad (n = 0, 1, 2, ...). \tag{1}$$

In the present paper the question of the product of bases in an arbitrary domain is studied for a broad class of bases, and it is possible to obtain a number of more precise results than in the papers mentioned. In addition, for arbitrary domains we consider questions concerning inverse bases and the quotient of two bases.

Let D be an open simply connected domain of the complex plane. In what follows we shall consider the following spaces of analytic functions: 1) $\mathfrak{A}(D)$–the space of functions analytic in D, with the topology of uniform convergence inside D; 2) $\mathfrak{A}(\overline{D})$–the space of functions analytic in $\overline{D}$, with the topology of the union of spaces ([9], p. 89):

$$\mathfrak{A}(\overline{D}) = \bigcup_{\lambda} \mathfrak{A}(D_\lambda),$$

where D_λ are open domains, with $D_\lambda \supset \overline{D}_{\lambda+1}$ and

$$\bigcap_{\lambda} \overline{D}_\lambda = \overline{D}, \quad \lambda = 1, 2, \dots .$$

A sequence $x_n(z)$ converges to $x(z)$ in $\mathfrak{A}(\overline{D})$ if and only if there exists λ_0 such that $x_n, x \in \mathfrak{A}(D_{\lambda_0})$ and $x_n \to x$ in $\mathfrak{A}(D_{\lambda_0})$. If the boundary of the domain D is a rectifiable Jordan curve, we shall also consider one more space: 3) $\mathfrak{A}C(D)$ —the space of functions analytic in D and continuous in $\overline{D}$, with norm

$$\|x\| = \max_{z \in \overline{D}} |x(z)|.$$

In the case when $D = K_r = \{|z| < r\}$, preserving the notation adopted in ([8]), we shall denote $\mathfrak{A}(D)$, $\mathfrak{A}(\overline{D})$, $\mathfrak{A}C(D)$ respectively by A_s, $\overline{A}_r$, $A_r C$.

Definition 1. Let E be a linear topological space and let $E_1 \subset E$. Then we shall call $x_n \in E$ a **basis in the topology** E **for** E_1 if every element $x \in E_1$ can be uniquely expanded in the series

$$x = \sum \xi_n x_n, \tag{2}$$

convergent in the topology of the space E. In the case $E_1 = E$ we obtain the usual Schauder basis in E.

Definition 2. The system x_n will be called a **pseudobasis in the topology** E **for** E_1 if every function from E_1 can be expanded in the series (2) (not necessarily uniquely).

* Here and below, the results of the cited papers are stated in the terms and notation adopted in ([8]).

Let Γ be a closed Jordan curve and

$$t = \Phi(z) = z + a_0 + a_1/z + \cdots, \qquad \Phi'_\infty(z) \neq z \tag{3}$$

a conformal mapping of the exterior of the curve Γ onto the exterior of the disk $K_\gamma = \{|t| \leq \gamma\}$. The images of the circles $\{|t| = r\}$ $(r > \gamma)$ under this mapping will be denoted by Γ_r, and the domains interior to Γ and Γ_r will be denoted respectively by D and D_r. We shall also denote

$$\beta = \inf\{r : D_r \supset K_\gamma\}, \qquad \alpha = \max_{|z|\in\Gamma_\beta} |z|. \tag{4}$$

Lemma 1. The inequality $\gamma < \beta < \alpha^*$ holds.

Consider the Faber polynomials $\Phi_n(z)$ for the contour Γ ([10]). Obviously, $\Phi_n(z)$ is a basis in $\mathfrak{A}(\overline{D})$, $\mathfrak{A}(D_r)$, $\mathfrak{A}(\overline{D}_r)$ $(r > \gamma)$.

Definition 3. A basis $p_n(z)$ in $\mathfrak{A}(\overline{D})$, or in $\mathfrak{A}(D_r)$, $\mathfrak{A}(\overline{D}_r)$, will be called **quasipower** if the series $\sum_n \xi_n p_n(z)$ and $\sum_n \xi_n \Phi_n(z)$ simultaneously converge or diverge in $\mathfrak{A}(\overline{D})$ (respectively in $\mathfrak{A}(D_r)$ and $\mathfrak{A}(\overline{D}_r)$) for every numerical sequence ξ_n**.

Definition 4. A basis $p_n(z)$ in $\mathfrak{A}(\overline{D})$ will be called **(quasipower-) continuable** to $\overline{D'}$ if $p_n(z)$ is a (quasipower) basis in $\mathfrak{A}(\overline{D'})$. If, moreover, $D' \subset D$ $(D \subset D')$, then $p_n(z)$ will be called an **internally (externally) continuable basis** (cf. ([11])).

The product of the systems $p_n^{(1)}(z)$ and $p_n^{(2)}(z)$ will henceforth be denoted by $\{p_n(z)\} = \{p_n^{(1)}(z)\}\{p_n^{(2)}(z)\}$. We formulate the main theorem.

Theorem 1. Let $p_n^{(1)}(z)$ be a basis in $\mathfrak{A}(\overline{D})$, continuable to $\overline{D}_\beta$, and let $p_n^{(2)}(z)$ be a quasipower basis in $\mathfrak{A}(\overline{D})$, quasipower-continuable to $\overline{D}_\alpha$***. Then the product

$$\{p_n(z)\} = \{p_n^{(1)}(z)\}\{p_n^{(2)}(z)\}$$

is a basis in the topology $\mathfrak{A}(\overline{D})$ for $\mathfrak{A}(\overline{D}_a)$ (for the definition of α and β see (4)).

Before passing to the proof of the theorem, we consider lemmas.

Lemma 2. In order that a basis $p_n(z)$ in $\overline{A}_\gamma$ be quasipower, it is necessary and sufficient that, for every r satisfying $\gamma < r < r_0$, there exist numbers $C'(r)$, $C''(r)$, $\rho_1(r)$, $\rho_2(r)$ such that

$$C'(r)\rho_1^n(r) \le \max_{|z|<r} |p_n(z)| \le C''(r)\rho_2^n(r), \tag{5}$$

where $\rho_i(r) \downarrow \gamma$ as $r \downarrow \gamma$.

For A_γ an analogous assertion was proved by Yu. F. Korobeinik.

Lemma 3. Let

$$x(z) = \sum_{n=0}^{\infty} \xi_n z^n$$

be a function analytic in $|z| \le r + \varepsilon$. Then

$$\sum_{n=0}^{\infty} |\xi_n| \, r^n \le \frac{r+\varepsilon}{\varepsilon} \max_{|z|\le r+\varepsilon} |x(z)|. \tag{6}$$

Lemma 4. Let

$$x(z) = \sum \xi_n z^n \in \overline{A}_\gamma.$$

Then the mapping $\mathfrak{R}: \ A_\gamma \to \mathfrak{A}(\overline{D})$ such that

$$y(z) = \mathfrak{R}x(z) = \sum \xi_n \Phi_n(z)$$

is an isomorphism of $\overline{A}_\gamma$ $(A_r, \overline{A}_r, \ r > \gamma)$ onto $\mathfrak{A}(\overline{D})$ (respectively onto $\mathfrak{A}(\overline{D}_r)$, $\mathfrak{A}(D_r)$, $r > \gamma$).

This assertion is obtained by applying the estimates for Faber polynomials (([10]), $p.$421) and Lemma 3.

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* It is assumed that Γ is different from a circle with center at the origin.
** For the disk this definition coincides with that adopted in ([8,11]).
*** Since, by (4), $D_\alpha \supset D_\beta \supset K_\gamma$, the $p_n^{(s)}(z)$ are analytic in $K_\gamma = \{|z| < \gamma\}$, $s = 1, 2$.

Corollary 1. If

$$p_n(z) = \sum q_{n,k} \Phi_k(z) \tag{7}$$

is a (quasi-power) basis in $\mathfrak{A}(\overline{D})$ (or in $\mathfrak{A}(D_r), \mathfrak{A}(\overline{D}_r)$), then

$$q_n(z) = \sum q_{n,k} z^k \tag{8}$$

is a (quasi-power) basis in $\overline{A}_\gamma$ (respectively in $A_r, \overline{A}_r$), and conversely.

Corollary 2. If (7) is a basis in the topology $\mathfrak{A}(\overline{D})$ for $\mathfrak{A}(\overline{D}_r)$, then (8) is a basis in the topology $\overline{A}_\gamma$ for $\overline{A}_r$, and conversely.

Let us proceed to the proof of the theorem. Let (7),

$$p_n^{(1)}(z) = \sum p_{n,k}^{(1)} z^k, \qquad p_n^{(2)}(z) = \sum q_{n,k}^{(2)} \Phi_k(z) \tag{9}$$

be the bases given by the conditions of the theorem, and let $q_n^{(2)}(z) = \sum q_{n,k}^{(2)} z^k$. In view of (7) and (9),

$$p_n(z) = \sum_i p_{n,i}^{(1)} p_i^{(2)}(z) = \sum_k \left(\sum_i p_{n,i}^{(1)} q_{i,k}^{(2)} \right) z^k = \sum_k q_{n,k} \Phi_k(z). \tag{10}$$

The order of summation in (10) may be interchanged, since, using the estimates for Faber polynomials and (5), (6), we have

$$\sum_i |p_{n,i}^{(1)}| \sum_k |q_{i,k}^{(2)}| \max_{z\in\Gamma_r} |\Phi_k(z)| \le M \sum_i |p_{n,i}^{(1)}| \sum_k |q_{i,k}^{(2)}| r^k \le$$

$$\le M \sum_i |p_{n,i}^{(1)}| \frac{r+\varepsilon}{\varepsilon} \max_{|z|\le r+\varepsilon} |q_i^{(2)}(z)| \le M \frac{r+\varepsilon}{\varepsilon} C''(r+\varepsilon) \sum_i |p_{n,i}^{(1)}| \rho_2^i(r+\varepsilon).$$

Taking into account that $p_n^{(1)}(z)$ are functions analytic in $\overline{D}_\beta$, and hence analytic in $A_{\gamma+\delta}$ for some $\delta > 0$, for r and ε such that $\rho_2(r+\varepsilon) \le \gamma + \delta$, we obtain the inequality

$$\sum_i |p_{n,i}^{(1)}| \sum_k |q_{i,k}^{(2)}| \max_{z\in\Gamma_r} |\Phi_k(z)| < M_n < \infty,$$

which proves the legitimacy of the interchanges in (10). From (10),

$$q_n(z) = \sum_k \left(\sum_i p_{n,i}^{(1)} q_{i,k}^{(2)} \right) z^k = \sum_i p_{n,i}^{(1)} q_i^{(2)}(z). \tag{11}$$

By Corollary 2, in order to prove the theorem it is enough to show that $q_n(z)$ is a basis in the topology $\overline{A}_\gamma$ for $\overline{A}_\alpha$. We first show that $p_n^{(1)}(z)$ is a basis in the topology $\overline{A}_\gamma$ for $\overline{A}_\alpha$. By assumption, $p_n^{(1)}(z)$ is a basis in $\mathfrak{A}(\overline{D}_\beta)$, and therefore every function from $\mathfrak{A}(\overline{D}_\beta)$ can be expanded in a series converging in $\mathfrak{A}(\overline{D}_\beta)$; since, by (3), $\overline{K}_\gamma \subset \overline{D}_\beta \subset \overline{K}_\alpha$, it is all the more true that every function from $\overline{A}_\alpha$ can be expanded in a series in $p_n^{(1)}(z)$, converging in $\overline{A}_\gamma$. The expansion will be unique. Indeed, suppose

$$\sum \xi_n p_n^{(1)}(z) = 0, \qquad \xi_n \ne 0, \tag{12}$$

where the series converges in $\overline{A}_\gamma$, i.e. converges in the disk $|z| < r_0$ $(r_0 > \gamma)$. From the continuation of $p_n^{(1)}(z)$ from $\overline{D}$ into $\overline{D}_\beta$ it follows that $p_n^{(1)}(z)$ is a basis in $\mathfrak{A}(D_\beta)$, moreover internally continuable (see (1)). But then, by (11), from the convergence of the series (12) inside the domain $|z| < r_0$, which has a boundary point on the boundary of the domain D_β, it follows that the series (12) converges in $\mathfrak{A}(D_\beta)$. The latter contradicts the fact that $p_n^{(1)}(z)$ is a basis in $\mathfrak{A}(D_\beta)$. Thus, $p_n^{(1)}(z)$ is a basis in the topology $\overline{A}_\gamma$ for $\overline{A}_\alpha$.

By Corollary 1, $q_n^2(z)$ is a quasi-power basis both in $\overline{A}_\gamma$ and in $\overline{A}_\alpha$. Then the mapping $\mathfrak{K}_1$, defined as follows: if $x(z) = \sum \xi_n z^n$, then $\mathfrak{K}_1 x = \sum \xi_n q_n^{(2)}(z)$, is an isomorphism of $\overline{A}_r$ onto itself $(\gamma \le r \le \alpha)$. Therefore, by the preceding remark and (11), $q_n(z)$ is a basis in the topology $\overline{A}_\gamma$ for $\overline{A}_\alpha$. The theorem is proved.

Theorem 1 contains Newns' s result ([7]), if the notation and terminology are brought into agreement. Namely, in ([7]) the assertion of Theorem 1 is proved for bases of the form (1). But such bases are quasipower-extendable into any $\mathfrak{A}(\overline{D}_r)$, $r > \gamma$, and hence certainly satisfy the conditions of Theorem 1.

The following theorem shows that the result of Theorem 1 is sharp for any pair of bases, in contrast to ([6], [7]), where it is proved that the result is sharp only for the whole class of simple bases.

Theorem 2. *The constant α in the conditions of Theorem 1 is sharp for every pair of bases $p_n^{(1)}(z)$ and $p_n^{(2)}(z)$, i.e. it cannot be decreased (and thereby the class of expandable functions enlarged).*

In what follows we shall need the following.

Theorem 3. *Let Γ be a regular curve. If $p_n(z)$ is a simple pseudobasis in the topology $\mathfrak{A}C(D)$ for $\mathfrak{A}(\overline{D})$, then $p_n(z)$ is a simple basis in $\mathfrak{A}(\overline{D})$, and conversely.*

The theorem is true when D is a disk with center at the origin (([2]), p. 462). Since the mapping $\mathfrak{R}$, considered in Lemma 4, is an isomorphism of $\overline{A}_\gamma$ onto $\mathfrak{A}(\overline{D})$ and, obviously, takes a simple basis into a simple one, the assertion of the theorem follows from the following lemma.

Lemma 5. *In the case of a regular contour Γ, the mapping $\mathfrak{R}$, considered in Lemma 4, is an isomorphism of $A_\gamma C$ onto $\mathfrak{A}C(D)$.*

From Theorems 1 and 3 there follows the result of ([6]), which we formulate differently from the cited paper.

Theorem 4. *Let Γ be a regular curve and let $p_n^{(i)}(z)$, $i = 1, 2$, be simple unit pseudobases in the topology $\mathfrak{A}C(D)$ for $\mathfrak{A}(\overline{D})$; then the system*

$$\{p_n(z)\} = \{p_n^{(1)}(z)\}\{p_n^{(2)}(z)\} \tag{13}$$

is a pseudobasis in the topology $\mathfrak{A}C(D)$ for $\mathfrak{A}(\overline{D}_\alpha)$.

Remark 1. We obtain here even more: the system (13) is a basis in the topology $\mathfrak{A}(\overline{D})$ for $\mathfrak{A}(\overline{D}_\alpha)$. This shows that the result of Barsoum and Nassif is in fact identical with Newns' s result. In particular, it follows from what has been set forth that the constants γ/α of Newns ([7]) and $\beta/S(\beta)$ of Nassif–Barsoum ([6]) coincide.

Let $p_n(z)$ be a basis in $\mathfrak{A}(\overline{D})$. We shall call the system $\{\pi_n(z)\} = \{p_n(z)\}^{-1}$ inverse to $p_n(z)$ if

$$\{\pi_n(z)\}\{p_n(z)\} = \{p_n(z)\}\{\pi_n(z)\} = \{z^n\}.$$

Let $z = \Psi(t)$ be the mapping inverse to the mapping (3), $t = \Phi(z)$. Put

$$\tau = \sup_{|t|=\gamma} |\Psi(t)| = \sup_{t\in\Gamma'} |t|.$$

Let Γ' be the preimage of the circle $|z| = \tau$ under the mapping $\Psi(t)$, and

$$\mu = \sup_{|z|=\tau} |\Phi(z)| = \sup_{t\in\Gamma'} |t|.$$

Theorem 5. *Let $p_n(z)$ be a quasipower basis in $\mathfrak{A}(\overline{D})$, quasipower-extendable into $\overline{D}_\mu$. Then the inverse system $\pi_n(z)$ is a basis in the topology $\overline{A}_\gamma$ for $\overline{A}_\mu$.*

In conclusion we give a theorem for the quotient of two bases in $\mathfrak{A}(\overline{D})$.

Theorem 6. *Let $p_n^{(1)}(z)$ be a basis in $\mathfrak{A}(\overline{D})$, and let $p_n^{(2)}(z)$ be a quasipower basis in $\mathfrak{A}(\overline{D})$. Then the quotient $\{p_n(z)\} = \{p_n^{(1)}(z)\}\{p_n^{(2)}(z)\}^{-1}$ is a basis in $\overline{A}_\gamma$.*

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REFERENCES

1. J.-M. Whittaker, *Sur les séries de base de polynomes quelconques*, Paris, 1949.
2. W. F. Newns, *Phyl. Trans. Roy. Soc. London*, A, **245**, 429 (1953).
3. W. F. Newns, *Proc. Roy. Soc.*, A, **237**, No. 1208, 55 (1956).
4. M. Nassif, *Am. J. Math.*, **79**, No. 4, 943 (1957).
5. M. Nassif, *Proc. Koninkl. Nederl. Akad. Wet.*, A, **60**, 598 (1957).
6. F. R. Barsoum, M. Nassif, ibid., A, **63**, No. 3, 333 (1960).
7. W. F. Newns, ibid., A, **63**, No. 2, 187 (1960).
8. M. G. Khaplanov, DAN, **80**, No. 2, 177 (1951).
9. I. M. Gelfand, G. E. Shilov, *Spaces of Basic and Generalized Functions*, vol. 2, Moscow, 1958.
10. A. I. Markushevich, *Theory of Analytic Functions*, Moscow-Leningrad, 1950.
11. M. M. Dragilev, *Matem. sbornik*, **53**, 2, 207 (1961).

Note: Figure translations are in progress. See original paper for figures.