V. V. VISHIN

1963

SovietRxiv

Abstract

Full Text

V. V. VISHIN

IDENTICAL TRANSFORMATIONS IN FOUR-VALUED LOGIC

(Presented by Academician P. S. Novikov on 8 I 1963)

The problem of identical transformations of formulas of k-valued logics reduces to finding complete systems of identities for closed classes of functions having a finite basis. R. C. Lyndon ([1]) showed that for every closed class of two-valued logic there exists a finite complete system of identities. He also ([2]) obtained a negative result, consisting in the fact that for $k = 7$ a closed class was constructed that has no finite system of identities. The question naturally arises: what is the least value of closed classes having no finite complete system of identities? In the present paper an example is constructed of a 4-valued closed class with this property, and therefore, for a complete solution of the question, it remains only to consider the case $k = 3$.

Let us consider the closed class of functions generated by the function xy, defined by Table 1. We shall denote this class by M. The function (xx), identically equal to zero, we shall denote by 0.

Table 1

$y\backslash x$	0	1	2	3
0	0	0	0	0
1	0	0	0	3
2	0	1	0	1
3	0	0	0	0

The notation $f = g$ will mean that the functions f and g are identically equal (i.e., have identical tables). Such notations we shall call **identities**, and in this case the formulas f and g will be called **equivalent**.

Lemma 1. In the class M the following identities hold:

$$A_1:\ (x0) = 0.$$

$$A_2:\ (0x) = 0.$$

$$A_3: \ ((x_1x_2)x_2) = (x_1x_2).$$

$$A_4: \ (x_1(x_2(x_3x_4))) = 0.$$

$$B_n: \ ((\dots(((x_1x_2)x_3)x_4)\dots x_n)x_2) = ((\dots(((x_1x_3)x_2)x_4)\dots x_n)x_3).$$

$$C_{n,k}: \ ((((\dots(((x_1x_2)x_3)x_4)\dots x_k)x_{k+1})\dots x_n)x_2)$$
$$= ((((\dots(((x_1x_2)x_3)\dots x_{k+1})x_k)\dots x_n)x_2) \quad (k = 4, 5, \dots, n).$$

$$D_n: \ ((\dots((x_1x_2)x_3)\dots x_n)x_1) = 0.$$

The validity of this lemma is easily established by direct verification.

Formulas in which all left parentheses stand to the left of all occurrences of variables we shall call **formulas of left association** and denote by φ; the remaining formulas we shall call **formulas of right association** and denote by $\overline{\varphi}$.

Lemma 2. If $\widehat{\varphi}_1 \neq 0$ and $\overline{\varphi}_2 \neq 0$, then in M it cannot be the case that $\widehat{\varphi}_1 = \overline{\varphi}_2$.

Proof. 1. We shall show that if $\widehat{\varphi}_1 = \overline{\varphi}_2$, then all variables occurring in the formula $\overline{\varphi}_2$ occur in $\widehat{\varphi}_1$. Suppose this is not so. Let $\widehat{\varphi}_1$ not depend on some x_k, while $\overline{\varphi}_2$ does depend on it; then set $x_k = 0$. By virtue of A_1 and A_2, $\overline{\varphi}_2 = 0$ for any values of the remaining variables, i.e. $\widehat{\varphi}_1 = 0$, which contradicts the assumption.

2. Suppose that $\hat{\varphi}_1 = \overline{\varphi}_2$. Let $\hat{\varphi}_1$ have the form

   $$((\cdots((x_{i_1}x_{i_2})x_{i_3})\cdots)x_{i_k}).$$

   Since $\hat{\varphi}_1 \neq 0$, by virtue of D_n and A_2, $i_1 \neq i_m$ $(m = 2, 3, \dots, k)$. Let $\overline{\varphi}_2$ have the form $(x_{j_1}\dots(y_1y_2)\dots)$, where y_2 is either a variable x_{i_r} $(1 \leq r \leq k)$, or some formula. Put $x_{i_1} = 3$, and the remaining $x_s = 1$; then $\hat{\varphi}_1 = 3$, and since $y_3 \neq 2$ and, consequently, $(y_1y_2) \neq 1$ and $(y_1y_2) \neq 2$, we have $\overline{\varphi}_2 = 0$, which contradicts the supposition.

Let a system of identities $\varphi_1 = \psi_1$, $\varphi_2 = \psi_2, \dots, \varphi_m = \psi_m$ be given. By an identical transformation of a formula with respect to this system we shall mean the replacement of one of its subformulas φ_i' by the subformula ψ_i', or conversely, where the identity $\varphi_i' = \psi_i'$ is obtained from $\varphi_i = \psi_i$ by substituting, in place of variables, arbitrary functions of the class under consideration or other variables. An identity $g_1 = g_2$ is derivable from the given system of identities $\varphi_1 = \psi_1, \dots, \varphi_m = \psi_m$ if, by identical transformations with respect to this system, g_1 can be obtained from g_2, or conversely, g_2 from g_1. A system of identities is called **complete** if every identity holding in the class of functions under consideration is derivable from it. Let us single out from M the subclass M' of all left-associative functions not identically equal to zero.

Suppose that in M there exists a complete finite system of identities $\varphi_1 = \psi_1, \dots, \varphi_m = \psi_m$; then, by Lemma 2, among them there are identities $\hat{\varphi}_{i_1} = \hat{\psi}_{i_1}, \dots, \hat{\varphi}_{i_k} = \hat{\psi}_{i_k}$. As a consequence of Lemma 2, the system $\hat{\varphi}_{i_1} = \hat{\psi}_{i_1}, \dots, \hat{\varphi}_{i_k} = \hat{\psi}_{i_k}$ must be complete in M', since an arbitrary $\hat{\varphi} = \hat{\psi}$ can be derived only from the system $\hat{\varphi}_{i_1} = \hat{\psi}_{i_1}, \dots, \hat{\varphi}_{i_k} = \hat{\psi}_{i_k}$.

Thus, if there exists a finite complete system of identities in M, then there also exists a finite complete system of identities in M'. We shall show that this necessary condition is not fulfilled; thereby it will be proved that in M there is no finite complete system of identities.

The system of identities $A_3,\ B_n,\ C_{n,k}\ (k = 4, \dots, n;\ n = 1, 2, \dots, m)$ in the class M' will be denoted by E_m. Since only left-associative formulas enter M', in writing them we shall omit all parentheses; in such formulas, subformulas of the form $x_i x_{j_1} \dots x_{j_k} x_i$, where $k > 0$, and also a single occurrence of a variable, we shall call **intervals**.

We shall call two intervals **nonintersecting** if they have no identical variables. If the variables entering an interval stand in increasing order of indices, except for its right end, then the interval will be called **normal**.

A collection of pairwise nonintersecting normal intervals will be called a **canonical form of a formula**.

Lemma 3. *For every formula there exists at most one canonical formula equivalent to it.*

Proof. Let there be two canonical formulas

$$\varphi_1 \sim y_1^1 \dots y_{m_1}^1 \quad \text{and} \quad \varphi_2 \sim y_1^2 \dots y_{m_2}^2,$$

where $y_i^1 \dots y_{m_i}^i$ are pairwise nonintersecting normal intervals, $i = 1, 2$, and suppose that $\varphi_1 = \varphi_2$.

1) We shall show that $y_1^1 \sim y_1^2$. Indeed, in view of D_n, the intervals y_1^1 and y_1^2 are variables; let, for example, $y_1^1 \sim x_i$. Putting $x_i = 3,\ x_j = 1$ for $j \neq i$, we obtain that $\varphi_1 = 3$, and if y_1^2 differs from x_i, then $\varphi_2 = 0$, which contradicts the supposition.

2) Suppose that $y_s^1 = y_s^2$ for $s = 1, \dots, k,\ k < \min(m_1, m_2)$, and show that then $y_{k+1}^1 \sim y_{k+1}^2$. Assume the contrary, i.e. that y_{k+1}^1 is not $\sim y_{k+1}^2$ (where one of these intervals may be empty). Then, because the intervals y_{k+1}^1 and y_{k+1}^2 are normal, in one of them, say in y_{k+1}^1, there will be a variable x_j not entering the interval y_{k+1}^2.

Since functions from M' depend essentially on all their variables, this means, in particular, that $k+1 < m_2$. Set the values of all variables from $y_2^2 \dots y_{k+1}^2$ equal to 1, and set the values of all remaining variables equal to 2. Then, evidently, $\varphi_2 = 1$, while $\varphi_1 = 0$, which contradicts the assumption that the functions φ_1 and φ_2 are equal.

Lemma 4. *Every identity of formulas having no more than m occurrences of variables (of length m) is derivable from the system E_m.*

It is easy to see that, by means of the system E_m, any function of length no more than m can be brought to canonical form; namely, by means of $C_{n,k}$ and B_n, intersecting intervals can be transformed into one, possibly longer, interval, and by means of A_3 and $C_{n,k}$ $(k = 4, \dots, n;\ n = 1, \dots, m)$, already nonintersecting intervals can be brought to normal form. As a consequence of Lemma 3, we obtain the validity of this lemma as well.

Theorem. *There is no finite complete system of identities in M.*

As indicated above, a necessary condition for the existence of a finite complete system of identities in M is the existence of such a system in M'.

Suppose that there exists a finite complete system of identities in M'. Let all the identities of this system contain formulas of length no more than m. Then, by Lemma 4, E_m is a complete system for M'. We shall show that B_{m+1} is not derivable from E_m, thereby proving the theorem.

Indeed, to either side of the identity B_{m+1}, from the system E_m only the identity A_3 is applicable. Likewise, only the identity A_3 from the system E_m is applicable to the result, since this result is not any substitution instance of the remaining identities from E_m. Thus, E_m is not complete. Consequently, the theorem is proved.

Received14 XII 1962

REFERENCES

[1] R. K. Lyndon, *Cybernetics Collection*, 1, IL, 1950, p. 246. [2] R. K. Lyndon, Identities in finite algebras, ibid., p. 249.

Note: Figure translations are in progress. See original paper for figures.