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1963

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Abstract

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ON THE SOLUTION OF CERTAIN BOUNDARY-VALUE PROBLEMS BY THE FOURIER METHOD

(Presented by Academician I. M. Vinogradov, 12 IV 1963)

Let l_θ be a (non-self-adjoint) boundary-value problem on the half-axis $R^+ = [0, \infty)$, generated by the differential expression $l[y] = -y'' + p(x)y$ and the boundary condition $y'(0) - \theta y(0) = 0$. To the boundary-value problem l_θ there corresponds the so-called l_θ-Fourier transform. In our papers ([1],[2]), under the assumption that for some $\varepsilon > 0$ the function $|p(x)| \exp \varepsilon x$ is summable on the half-axis R^+, the l_θ-Fourier transform was extended to a certain class of exponentially growing functions. It is of interest to ask whether the theory of the l_θ-transform can be used to study the corresponding boundary-value problems by the Fourier method in some class of unbounded functions. Especially interesting is the case where the boundary-value problem l_θ has so-called spectral singularities (see ([2])), since in this case the l_θ-transform does not admit a unique inversion. From what follows we shall see that even in the presence of spectral singularities, with the aid of the l_θ-transform one can obtain fairly concrete results, among them the existence, uniqueness, explicit representation, and also estimates of solutions of the boundary-value problems under consideration. In the present paper we adhere to the same terminology and notation as in ([1],[2]).

For each real η, by $L^2_\eta(R^+)$ we denote the Hilbert space corresponding to the norm

$$\|h\|_\eta = \left\{ \int_0^\infty |h(x)e^{\eta x}|^2 \, dx \right\}^{1/2}, \tag{1}$$

and put

$$\Phi = \bigcap_{0<\eta<\varepsilon_0} L^2_{+\eta}(R^+), \qquad F = \bigcup_{0<\eta<\varepsilon_0} L^2_{-\eta}(R^+); \tag{2}$$

here ε_0 is a positive number depending on the boundary-value problem l_θ, whose definition is indicated in ([2]). We regard the set Φ as a countably Hilbert space with the system of norms $\|\cdot\|_\eta$, $0 < \eta < \varepsilon_0$, and the set F as the linear space conjugate to the space Φ: $F = \Phi'$. Accordingly, we shall write

$$\int_0^\infty \varphi(x) f(x)\, dx = \langle \varphi, f \rangle, \qquad \varphi \in \Phi, \quad f \in F. \tag{3}$$

One may say that Φ is the space of functions decreasing faster than $\exp(-\varepsilon_0 x)$ as $x \to \infty$, and F is the space of functions increasing more slowly than $\exp \varepsilon_0 x$ as $x \to \infty$. Let us define the operators generated by the differential expression $l[y]$ and the boundary condition $y'(0) = \theta y(0)$ in the spaces under consideration. Denote by $\mathfrak{D}_\eta(L)$ the set of functions h which have a derivative h' absolutely continuous on every finite interval of the half-axis R^+, satisfy the boundary condition $h'(0) = \theta h(0)$, and are such that $h \in L^2_\eta(R^+)$ and $l[h] \in L^2_\eta(R^+)$. Put

$$\mathfrak{D}_\Phi(L) = \bigcap_{0<\eta<\varepsilon_0} \mathfrak{D}_{+\eta}(L), \qquad \mathfrak{D}_F(L) = \bigcup_{0<\eta<\varepsilon_0} \mathfrak{D}_{-\eta}(L). \tag{4}$$

and $Lh = l[h]$ for $h \in \mathfrak{D}_\eta(L)$, $|\eta| < \varepsilon_0$. Let $f, g \in F$. It can be shown that $f \in \mathfrak{D}_F(L)$ and $Lf = g$ if and only if for all $\varphi \in \mathfrak{D}_\Phi(L)$ the relation $\langle L\varphi, f \rangle = \langle \varphi, g \rangle$ holds.

Using the theory of the l_θ-Fourier transform, one can construct "functions" of the operator L in the space F. We shall say that a function $\mathfrak{F}(\lambda)$ of the complex variable λ belongs to the class $(\mathcal{L})$ if it has the following properties: a) the domain of definition of the function $\mathfrak{F}(s^2)$ contains the strip $|\operatorname{Im} s| < \varepsilon_0$; b) the restriction of $\mathfrak{F}(s^2)$ to the strip $|\operatorname{Im} s| < \varepsilon_0$ is a holomorphic function of s, satisfying the condition

$$|\mathfrak{F}|_{\eta k} = \sup_{|\operatorname{Im} s| < \eta} \frac{|\mathfrak{F}(s^2)|}{1 + |s|^{2k}} < \infty \tag{5}$$

for some nonnegative integer k and for all η, $0 < \eta < \varepsilon_0$; c) the function $\mathfrak{F}(\lambda)$ is holomorphic in a neighborhood of the eigenvalues $\lambda_1, \dots, \lambda_r$ of the restriction of the operator L to the Hilbert space $L^2(R^+)$.

Theorem. There exists a correspondence $\mathfrak{F} \to \mathfrak{F}(L)$, assigning to each function $\mathfrak{F} \in (\mathcal{L})$ a linear operator $\mathfrak{F}(L)$ acting in the space F, such that the conditions 1°–4° listed below are satisfied.

1°. If $|\mathfrak{F}|_{\eta k} < \infty$, $0 < \eta < \varepsilon_0$, then*

$$\mathfrak{D}_{-\eta}(L^k) \subset \mathfrak{D}_{-\eta}(\mathfrak{F}(L)), \tag{6}$$

and there exists a constant $\delta(\eta)$ such that

$$\|\mathfrak{F}(L) f\|_{-\eta} \le \delta(\eta) \|\mathfrak{F}\|_{\eta k} \cdot \sum_{\nu=0}^{k} \|L^\nu f\|_{-\eta}, \tag{7}$$

where $\|\mathfrak{F}\|_{\eta k}$ is the largest of the three numbers**

$$|\mathfrak{F}|_{\eta k}, \qquad \max_{\substack{j=0,\dots,\mu_k-1\\ k=1,\dots,\rho}} |\mathfrak{F}^{(j)}(\tilde{\lambda}_k)|, \qquad \max_{\substack{j=0,\dots,m_k-1\\ k=1,\dots,r}} |\mathfrak{F}^{(j)}(\lambda_k)|. \tag{8}$$

In particular, if $|\mathfrak{F}|_{\eta 0} < \infty$, $0 < \eta < \varepsilon_0$, then the operator $\mathfrak{F}(L)$ is defined on the whole space F and is a continuous mapping of each of the Hilbert spaces $L^2_{-\eta}(R^+)$ into itself.

Relation (7) shows that the operator $\mathfrak{F}(L)$ depends on $\mathfrak{F}$ continuously in the sense of the norm $\|\cdot\|_{\eta k}$. The correspondence $\mathfrak{F} \to \mathfrak{F}(L)$ also has the following property of "weak" continuity:

2°. Let $\{\mathfrak{F}_t\}$ be a family of functions $\mathfrak{F}_t \in (\mathcal{L})$ such that: a) $|\mathfrak{F}_t|_{\eta k} < C_\eta < \infty$, $0 < \eta < \varepsilon_0$, where C_η is a constant independent of t; b) $\mathfrak{F}_t(\lambda) \to 0$ as $t \to 0$ uniformly in every compact part of the set $\{\lambda : \lambda = s^2,\ |\operatorname{Im} s| < \varepsilon_0\}$ and in some neighborhood of the points $\lambda_1, \dots, \lambda_r$. Then

$$\langle \varphi, \mathfrak{F}_t(L) f \rangle \underset{t\to 0}{\longrightarrow} 0 \quad \text{for all } \varphi \in \Phi,\ f \in \mathfrak{D}_F(L^k). \tag{9}$$

3°. If $\mathfrak{F}(\lambda) \equiv 1$, then $\mathfrak{F}(L)$ is the identity operator in the space F; if $\mathfrak{F}(\lambda) \equiv \lambda$, then $\mathfrak{F}(L) = L$.

4°. If $\mathfrak{F}, \mathfrak{G} \in (\mathcal{L})$, and α and β are arbitrary complex numbers, then

$$\alpha\mathfrak{F}(L) + \beta\mathfrak{G}(L) \subset (\alpha\mathfrak{F} + \beta\mathfrak{G})(L), \quad \mathfrak{F}(L)\mathfrak{G}(L) \subset (\mathfrak{F}\mathfrak{G})(L). \tag{10}$$

For lack of space we cannot give the proof of this theorem. Let us only note that, using the same notation as in formula (18) of article ([2]), we have

$$\mathfrak{F}(L)f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} s\omega(x,s^2)\mathfrak{F}(s^2)\frac{1}{A(s)A(-s)}\omega(f,s^2)\,ds +$$

* $\mathfrak{D}_{-\eta}(\bullet)$ denotes the domain of definition of the restriction of the operator $\bullet$ to the space $L^2_{-\eta}(R^+)$.

** μ_k denotes the multiplicity of the spectral singularity $\tilde{\lambda}_k$, and m_k the multiplicity of the eigenvalue λ_k ([2]).

$$+\sum_{k=1}^{\rho}\sum_{j=0}^{\mu_k-1} C_{kj}\left\{\left(\frac{d}{d\lambda}\right)^j \mathfrak{F}(\lambda)\omega(x,\lambda)\right\}_{\lambda=\tilde{\lambda}k} + \sum_{k=1}^{r}\left\{\left(\frac{d}{d\lambda}\right)^{m_k-1} M_k(\lambda)\mathfrak{F}(\lambda)\omega_f(\lambda)\omega(x,\lambda)\right\}_{\lambda=\lambda_k}. \tag{11}$$

Accordingly, $\mathfrak{F} \to \mathfrak{F}(L)$ can be used to construct solutions of various boundary-value problems in the quarter-plane $x > 0,\ t > 0$. Consider, for example, the boundary-value problem determined by the relations:

$$u_t = u_{xx} - p(x)u, \qquad (u_x - \theta u)\big|_{x=0} = 0, \qquad u\big|_{t=+0} = f(x). \tag{12}$$

It is not hard to see that the function $\mathfrak{F}_t(\lambda) = e^{-\lambda t}$ belongs to the class $(\mathcal{L})$ and satisfies condition (5) for $k = 0$ even after multiplication by any power of λ. Moreover, $\mathfrak{F}_t(\lambda) \to 1$ as $t \to 0$ with observance of the conditions indicated in Proposition 2°, and, for $t > 0$, $[\mathfrak{F}_{t+h}(\lambda) - \mathfrak{F}_t(\lambda)]/h + \lambda\mathfrak{F}_t(\lambda) \to 0$, as $h \to 0$, in the sense of the norm $\|\cdot\|_{\eta,0}$. From this it follows that, for every function $f \in L^2_{-\eta}(R^+)$, where $0 < \eta < \varepsilon_0$, the function

$$u = f(x,t) = e^{-tL} f(x) \tag{13}$$

belongs, for $t > 0$, to the domain of definition $\mathfrak{D}_{-\eta}(L^n)$ of any power L^n of the operator L and is (infinitely) differentiable with respect to t in the sense of the norm $\|\cdot\|_{-\eta}$. The function (13) satisfies the differential equation and the boundary condition (12). It satisfies the initial condition in the weak sense, i.e. $\langle\varphi, f(\cdot,+0)\rangle = \langle\varphi, f\rangle$ for all $\varphi \in \Phi$. If $f \in \mathfrak{D}_{-\eta}(L)$, then the limit $f(\cdot,+0) = f$ exists in the sense of the norm $\|\cdot\|_{-\eta}$. Finally, the function (13) admits an estimate of the form $\|f(\cdot,t)\|_{-\eta} \le C(\eta)e^{\eta^2 t}\|f\|_{-\eta}$.

Let us also consider the boundary-value problem

$$u_{tt} = u_{xx} - p(x)u, \qquad (u_x - \theta u)\big|_{x=0} = 0, \qquad u\big|_{t=+0} = 0, \qquad u_t\big|_{t=0} = f(x). \tag{14}$$

Applying arguments analogous to those used in the preceding example, we arrive at the following conclusions. The operator $\sin t\sqrt{L}/\sqrt{L}$ is defined on the whole space F, and for every function $f \in \mathfrak{D}_{-\eta}(L)$ the function

$$u = f(x,t) = \frac{\sin t\sqrt{L}}{\sqrt{L}}\, f(x) \tag{15}$$

belongs to $\mathfrak{D}_{-\eta}(L)^*$ and is twice differentiable with respect to t in the sense of the norm $\|\cdot\|_{-\eta}$. The function (15) is a solution of the boundary-value problem (14), which satisfies the initial condition in the sense of the norm $\|\cdot\|_{-\eta}$. The solution (15) also admits an estimate of the form $\|f(\cdot,t)\|_{-\eta} \le C(\eta)e^{\eta t}\|f\|_{-\eta}$. We note that the solutions we have constructed for the boundary-value problems (12) and (14) preserve the "integral" exponent η of exponential growth of the initial function f.

It is natural also to consider the boundary-value problem for an equation of Schrödinger type $u_t = i[u_{xx} - p(x)u]$. Here, however, a difficulty is encountered, consisting in the following: the function $\mathfrak{F}_t(\lambda) = e^{-i\lambda t}$ corresponding to the last equation does not belong to the class (L), since the function $e^{-is^2 t}$ has exponential order of growth with respect to s in the strip $|\operatorname{Im} s| < \varepsilon_0$, and consequently $\mathfrak{F}_t|_{\eta k} = \infty$ for arbitrarily large k (see relation (5)). We observe, however, that if in our considerations the space F of functions growing more slowly than $\exp \varepsilon_0 x$ is replaced by the space of functions of finite order of growth, then it will be possible to construct a correspondence $\mathfrak{F} \to \mathfrak{F}(L)$ in which the role of the strip $|\operatorname{Im} s| < \varepsilon_0$ is played by the real axis $\operatorname{Im} s = 0$, on which the function $e^{-is^2 t}$ is already bounded. This makes it possible to construct a solution of the boundary-value problem also for the equation $u_t = i[u_{xx} - p(x)u]$ with an initial sufficiently smooth function f of finite order of growth. Consequently, there is

* If $f \in \mathfrak{D}_{-\eta}(L^n)$, then also $f(\cdot, t) \in \mathfrak{D}_{-\eta}(L^n)$.

the well-known analogy between the boundary-value problems under consideration and the Cauchy problem for the corresponding equations with coefficients independent of x (see ([3])).

In the case when the boundary-value problem l_0 has no spectral singularities ([2]), the uniqueness of the solutions of the boundary-value problems (12) and (14) in the space F can be proved by the usual Holmgren method (see ([3])). We indicate a device by means of which uniqueness can be proved in the case when there are spectral singularities and, consequently, the l_0-Fourier transform is not uniquely invertible.

Denote by Φ_0 the set of functions $\varphi \in \Phi$ having the property that each of the spectral singularities $\tilde{\lambda}_1, \dots, \tilde{\lambda}_\rho$ of the boundary-value problem l_0 is a zero of multiplicity (at least) $\mu_1, \dots, \mu_\rho$, respectively, of the l_0-Fourier transform $\omega\varphi(\lambda)$ of the function φ.

Lemma. *In order that a function $f \in F$ satisfy the relation*

$$\langle \varphi, f \rangle = 0 \quad \text{for all } \varphi \in \Phi_0, \tag{16}$$

it is necessary and sufficient that *

$$f(x) = \sum_{k=1}^{\rho} \sum_{j=0}^{\mu_k - 1} C_{kj} \omega^{(j)}(x, \tilde{\lambda}_k), \tag{17}$$

where C_{kj} are certain constants.

Let now $f(x,t)$ be a solution of the boundary-value problem (12) corresponding to the initial function $f(x) \equiv 0$. We must show that if $f(\cdot, t) \in F$, then $f(x,t) \equiv$

0**. It can be proved that for an arbitrary function $\varphi \in \Phi_0$ the boundary-value problem (12) with initial function $f = \varphi$ has a solution $\varphi(x,t)$, which for all $t \geq 0$ also belongs to the set Φ_0. Consider an arbitrary $t > 0$ and, for $\tau \leq t$, put $\alpha(\tau) = \langle \varphi(\cdot, t-\tau), f(\cdot, \tau) \rangle$. We have $\alpha(0) = 0$ and, as is easy to see, $\alpha'(\tau) = 0$. Consequently, $\alpha(\tau) \equiv 0$ and, in particular, $\langle \varphi, f(\cdot, t) \rangle = 0$. On the basis of the preceding lemma we conclude from this that

$$f(x,t) = \sum_{k=1}^{\rho} \sum_{j=0}^{\mu_k - 1} C_{kj}(t) \omega^{(j)}(x, \tilde{\lambda}_k). \tag{17'}$$

Taking into account that the function (17′) satisfies the differential equation $u_t + l[u] = 0$, that $l[\omega^{(j)}(\cdot, \tilde{\lambda}_k)] = \tilde{\lambda}_k \omega^{(j)}(\cdot, \tilde{\lambda}_k) + j\omega^{(j-1)}(\cdot, \tilde{\lambda}_k)$, and that the functions $\omega^{(j)}(\cdot, \tilde{\lambda}_k)$ are linearly independent, we arrive at the conclusion that the functions $C_{kj}(t)$ satisfy a certain linear homogeneous system of (independent) differential equations. Since, moreover, $C_{kj}(0) = 0$ (in view of the initial condition $f(x, +0) = 0$), it follows that $C_{kj}(t) \equiv 0$, as was required to prove. An analogous device can be applied to prove uniqueness in the space F of the solution of the boundary-value problem (14). In this case it is expedient to define the function $\alpha(t)$ by the formula $\alpha(\tau) = \langle \varphi_t(\cdot, t-\tau), f(\cdot, \tau) \rangle + \langle \varphi(\cdot, t-\tau), f_t(\cdot, \tau) \rangle$, where $\varphi(x,t)$ is the solution of the boundary-value problem (14) corresponding to the initial function $f = \varphi \in \Phi_0$, such that $\varphi(\cdot, t) \in \Phi_0$.

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Received
4 IV 1963

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$*$ $\omega(x,\lambda)$ denotes the solution of the equation $l[y] = \lambda y$ with initial values $\omega(0,\lambda) = 1$, $\omega'_x(0,\lambda) = 0$; $\omega^{(j)}$ is the derivative of order j of ω with respect to the variable λ.

$**$ A. G. Kostyuchenko pointed out to the author that in fact uniqueness here holds in the class of functions growing as $x \to +\infty$ like $\exp(x^2)$.

Note: Figure translations are in progress. See original paper for figures.

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