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Abstract

Full Text

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ON SOME WAYS OF SPECIFYING SETS IN GENERATING BASES

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1. In the present note we use the introduction in (^{1})* of the concepts: **generating basis**, **calculus in a given generating basis**, **set strictly representable in a given generating basis**, etc.

If A is some alphabet, then by an A-**partition** we shall mean any ordered pair of solvable sets of A-words such that each member of the pair is the complement of the other member of the pair in the set of all A-words; an A-partition will be called **proper** if the first set of the pair is infinite. Let A be some alphabet, R some A-partition, and B some generating basis whose alphabet is A. Then:

a) the words belonging to the first set (the second set) of the A-partition R will be called R-**principal** words (respectively, R-**auxiliary** words);

b) if $\mathfrak{M}$ is some set of R-principal words, then we shall say that a calculus Ω in the basis B R-**represents** the set $\mathfrak{M}$, if in Ω all words of the set $\mathfrak{M}$ are derivable and every R-principal word derivable in Ω belongs to $\mathfrak{M}$;

c) if $\mathfrak{M}$ is some set of R-principal words, then we shall say that $\mathfrak{M}$ is R-**representable** in the basis B, if one can construct a calculus in the basis B which R-represents $\mathfrak{M}$;

d) the basis B will be called R-**sufficient** if every enumerable set of R-principal words is R-representable in B;

e) the basis B will be called **sufficient** if one can indicate a proper A-partition R' such that the basis B is R'-sufficient.

In the literature, various refinements of the general concept of a calculus have been proposed (see the bibliography in (^{1})), each of which is a definition of a calculus in some sufficient basis. At the same time, almost all these definitions are such that, for any proper A-partition R, one can construct an enumerable set of R-principal words that cannot be strictly represented by any calculus introduced by the definition under consideration (Theorem 1 of (^{1})). For example, for any alphabet A' one can construct an enumerable set of A'-words that cannot be strictly represented by any canonical calculus (^{2}) in the alphabet A'. At the same time, if A is any proper extension of the alphabet A' and R is the A-partition whose first set is the set of all A'-words**, then for any enumerable set $\mathfrak{M}$ of R-principal words one can construct a canonical calculus in the alphabet A that R-represents $\mathfrak{M}$.

2. We shall call a generating basis B **strongly solvable** if one can construct an algorithm which, for each scheme H of the basis B, each list of words $P_0, P_1, \dots, P_m$, where m is the index of the scheme H, and each of a list of numbers $i_0, i_1, \dots, i_m$ such that

$$\forall j\,[(0 \le j \le m) \supset (i_j = 0 \vee i_j = 1)]$$

recognizes whether it is possible to construct a list of words $Q_0, Q_1, \dots, Q_m$ such that Q_0 is derivable from the words $Q_1, Q_2, \dots, Q_m$ in one step of applying the scheme H and

$$\forall j\,[(0 \le j \le m) \supset ((i_j = 0 \supset Q_j = P_j)\ \&\ (i_j = 1 \supset Q_j > P_j))]\ {}^{*}.$$

From the strong decidability of a basis , obviously, weak decidability and decidability of follow. As a rule, the calculi used in mathematical logic are built in strongly decidable bases (in particular, the basis of canonical calculi is strongly decidable).

For any alphabet A, one can construct a strongly decidable enumerable basis , whose alphabet is A, a proper A-partition R, and a decidable set $\mathfrak{B}$ of R-auxiliary words such that, for any enumerable set $\mathfrak{M}$ of R-principal words, the set $\mathfrak{M} \mid \mathfrak{B}$ is strictly representable in . (As one may choose the basis of canonical calculi in the alphabet A, as R an A-partition whose first set is the set of words of the form PP, where P is an arbitrary A-word**, and as $\mathfrak{B}$ a decidable set of R-auxiliary words that are in constructive one-to-one correspondence with the set of all possible triples of the form $(S, \mathfrak{Q}, n)$, where S is an A-word, $\mathfrak{Q}$ is a canonical calculus in the alphabet A, n is a natural number, and S is derivable in $\mathfrak{Q}$ in no more than n steps.) On the other hand, the following holds.

Theorem 1. Whatever the alphabet A, the strongly decidable enumerable basis of generation , whose alphabet is A, and the decidable set $\mathfrak{B}$ of A-words having an infinite complement, one can construct a decidable infinite set $\mathfrak{M}$ of A-words such that the set $\mathfrak{M} \cap \mathfrak{B}$ is empty and the set $\mathfrak{M} \cup \mathfrak{B}$ is not strictly representable in the basis .

We outline the proof of the theorem. Let $H_1, H_2, \dots, H_n$ be some list of schemes of the basis ; m_i $(1 \le i \le n)$ the index of the scheme H_i; $\mathfrak{N}$ some finite set of A-words; N the greatest element of the set $\mathfrak{N}$; Ω_N the set of A-words greater

* In (1) the following misprints distorting the sense were allowed: 1) p. 272, line 24: instead of $(0 \leqslant \mu \leqslant n)$ one should read $(0 \leqslant \mu \leqslant m)$; 2) p. 273, lines 23 and 24: instead of $\mathfrak{B}$ one should read B; 3) p. 273, line 31: instead of one should read $\mathfrak{G}$.

** Such an A-partition will hereafter be denoted by $U_{A'}$.

than N; $\mathfrak{B}_N$ the intersection of Ω_N with the complement of the set $\mathfrak{B}$ to the set of all A-words; P_0 the least element of the set $\mathfrak{B}_N$.

It is possible algorithmically to recognize whether the condition is satisfied:

Whatever the scheme H_i $(1 \leq i \leq n)$ and the list of words $P_1, P_2, \dots, P_{m_i}$ belonging to the set $(\mathfrak{N} \cup \Omega_N) \setminus \{P_0\}$, the word P_0 is not derivable from the words $P_1, P_2, \dots, P_{m_i}$ in one step by H_i.

If this condition is not fulfilled, then one can construct a word P belonging to $\mathfrak{B}_N$, a scheme H_j $(1 \leq j \leq n)$, and a list of words $P_1, P_2, \dots, P_{m_j}$ belonging to the set $(\mathfrak{N} \cup \Omega_N) \setminus \{P\}$, such that: 1) P is derivable from the words $P_1, P_2, \dots, P_{m_j}$ in one step by H_j, and 2) whatever the word P' belonging to the set $\mathfrak{B}_N$ and not exceeding the greatest of the words $P, P_1, P_2, \dots, P_{m_j}$, the scheme H_i $(1 \leq i < j)$, and the list of words $P_1', P_2', \dots, P_{m_i}'$ belonging to the set $(\mathfrak{N} \cup \Omega_N) \setminus \{P'\}$, the word P' is not derivable from the words $P_1', P_2', \dots, P_{m_i}'$ in one step of applying H_i. Using this, one can give an inductive construction of the set whose existence is asserted by the theorem.

One can construct a weakly decidable and decidable enumerable basis $'$ and a decidable set $\mathfrak{B}$ having an infinite complement, such that every enumerable set containing $\mathfrak{B}$ is strictly representable in $'$.

3. If A and are some alphabets, then an **algorithm of type** $A \to$ will mean any algorithm that transforms every A-word to which it is applicable into a finite system of -words. If the indicated algorithm is applicable to every A-word, then we shall call it **complete**.

* The meaning of the expression $P > Q$, where P and Q are words, was explained in (1).

** Such an A-partition will henceforth be denoted by V_A.

Let F be some algorithm of type $A \Rightarrow B$. We shall say that a set $\mathfrak{N}$ of B-words **corresponds** to a set $\mathfrak{M}$ of A-words **by virtue of the algorithm** F, if $\mathfrak{N}$ has the following properties: 1) if $P \in \mathfrak{M}$, then one can construct a word Q such that Q is a member of the system $F(P)$ and $Q \in \mathfrak{N}$; 2) if $Q \in \mathfrak{N}$, then one can construct a word P such that Q is a member of the system $F(P)$, and every A-word S such that Q is a member of the system $F(S)$ belongs to $\mathfrak{M}$. Let F be some algorithm of type $A \Rightarrow B$; $\mathfrak{F}$ some set of algorithms of type $A \Rightarrow B$; B some generating basis whose alphabet is B. We shall say that an enumeration Ω in the basis B **strictly represents** a set $\mathfrak{M}$ of A-words **by means of** F, if Ω strictly represents some set $\mathfrak{N}$ of B-words corresponding to the set $\mathfrak{M}$ by virtue of F. We shall say that a set $\mathfrak{M}$ of A-words is **strictly representable in the basis** B **by means of a set of algorithms** $\mathfrak{F}$ (**by means of an algorithm** F, if F is the only element of the set $\mathfrak{F}$), if one can construct an enumeration Ω in the basis B and an algorithm F, belonging to $\mathfrak{F}$, such that Ω strictly represents $\mathfrak{M}$ by means of F.

The possibilities of strict representation of sets by means of algorithms are characterized by

Theorem 2. *Whatever the alphabet A, the nonempty enumerable set $\mathfrak{M}$ of A-words, and the generating basis B in which at least one finite set of B-words is strictly representable (B is the designation of the alphabet of the basis B), one can construct an algorithm F of type $A \Rightarrow B$ by means of which $\mathfrak{M}$ is strictly representable in B. If the indicated assumptions are satisfied and, in addition, either $\mathfrak{M}$ is decidable, or $\mathfrak{M}$ is infinite and the basis A is sufficient, then the aforementioned algorithm F can be constructed as a complete one.*

We construct the desired complete algorithm for the case of infinite $\mathfrak{M}$ and sufficient B. $\mathfrak{M}$ has an infinite decidable subset, which we denote by $\mathfrak{M}'$. One can construct a regular B-partition R such that the basis B is R-sufficient. Denote by G some complete algorithm of type $A \Rightarrow B$ that maps the set of all A-words one-to-one onto the set of R-basic B-words. The sets $\mathfrak{M}$ and $\mathfrak{M}'$ will be carried by this mapping into the sets $\mathfrak{N}$ and $\mathfrak{N}'$, respectively. One can construct a set $\mathfrak{W}$ of R-auxiliary words such that $\mathfrak{N} \cup \mathfrak{W}$ is strictly representable in B. Construct a complete algorithm of type $A \Rightarrow B$ which, on A-words not belonging to $\mathfrak{M}'$, coincides with G, and on elements of the set $\mathfrak{M}'$ is defined so that the set of its values coincides with the set $\mathfrak{N}' \cup \mathfrak{W}$. This algorithm will be the desired one.

Corollary. *If the hypotheses of Theorem 2 concerning the basis B are fulfilled, then for any enumerable set $\mathfrak{M}$ of A-words there cannot fail to exist an algorithm of type $A \Rightarrow B$ (if B is sufficient, then a complete algorithm of type $A \Rightarrow B$) by means of which $\mathfrak{M}$ is strictly representable in B.*

Remark. The notion of strict representability by means of algorithms generalizes the notion of strict representability proposed in (1). In a completely analogous way one can generalize the notion of R-representability. R-representability by means of algorithms extends the possibilities of representing sets by enumerations. For example, for every alphabet A' one can construct an enumerable set of A'-words which is not $U_{A'}$-representable by associative enumerations over the alphabet A; at the same time every enumerable set of A'-words is $U_{A' \cup \{\beta\}}$-representable by associative enumerations over the alphabet $A' \cup \{\beta\}$ by means of an algorithm of type $A' \Rightarrow A' \cup \{\beta\}$ that transforms every A'-word P into the word $\beta P \beta$ (see (3)). Another example is the $U_{\{\,\}}$-representability of sets of $\{\,\}$-words by means of an algorithm of type $\{\,\} \Rightarrow \{\,\}$ that converts every $\{\,\}$-word of length p into a $\{\,\}$-word of length 2^p. Every enumerable set of $\{\,\}$-words is $V_{\{\,\}}$-represented by means of this algorithm by canonical enumer-

in the alphabet $\{|\}$, each scheme of which has the form

$$\frac{q^n\ |^l}{q^{n'}\ |^{l'}},$$

(the simple representability of any enumerable set of $V_{\{|\}}$-basic words, obtained by Theorem 2 from (3), is effected by means of calculi of a more complicated structure).

4. The following theorems are proved analogously to Theorems 1 and 2 from (1) and are their strengthenings.

Theorem 3. *Whatever the alphabet A, the weakly decidable enumerable generation basis B (we denote its alphabet by B) and an algorithm F of type $A \Rightarrow B$, whose domain of applicability is infinite (an enumerable set $\mathfrak{F}$ of complete* algorithms of type $A \Rightarrow B$), one can construct an infinite decidable set of A-words which is not strictly representable in B by means of F (respectively, by means of $\mathfrak{F}$).*

We shall say that an algorithm F of type $A \Rightarrow B$ is **regular** if:

1) for every B-word Q one can construct an A-word P such that

$$\forall RS\ [(S \text{ is a member of the system } F(R)\ \&\ R > P) \supset S > Q]$$

(R is a variable for A-words, S is a variable for B-words);

2) for every A-word P one can construct a B-word Q such that

$$\forall RS\ [(S \text{ is a member of the system } F(R)\ \&\ P > R) \supset Q > S]\ **.$$

Theorem 4. *Whatever the alphabet A, the weakly decidable and decidable enumerable generation basis B (we denote its alphabet by B) and a regular algorithm F of type $A \Rightarrow B$, whose domain of applicability is infinite (an enumerable set $\mathfrak{F}$ of complete regular algorithms of type $A \Rightarrow B$), one can construct an infinite decidable set of A-words $\mathfrak{M}$ which restricts the decidability of the set strictly representable in B by means of F (respectively, by means of $\mathfrak{F}$).*

5. The class of sets strictly representable in a given basis is, generally speaking, not closed even under such operations as union and intersection of sets, projection of a set onto an alphabet, etc. For each of these operations it is not difficult to construct a generation basis and sets strictly representable in this basis such that the result of applying the operation to the corresponding sets is not strictly representable in this basis. For the last two operations the same assertion remains true also when passing to strict representability by means of an enumerable set of algorithms. Conversely, the operation of union is such that, for many bases, one can construct an algorithm of type $A \Rightarrow A$ (A is the alphabet of the corresponding basis), by means of which the union of any strictly representable sets of A-words is strictly representable. The same holds, for example, for the operations of "products," "iteration" (see (4), 7.1) and for any finite composition of them and the operation of union.

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CITED LITERATURE

1. S. Yu. Maslov, DAN, **152**, No. 2 (1963).
2. E. L. Post, *Am. J. Math.*, **65**, 197 (1943).
3. S. Yu. Maslov, DAN, **147**, No. 4 (1962).
4. S. K. Kleene, *Automata*, Moscow, 1956.

* Instead of completeness of all algorithms from $\mathfrak{F}$, it is sufficient to require the existence of a finite enumerable set of A-words on which all algorithms from $\mathfrak{F}$ are defined.

** For algorithms of type $A \Rightarrow B$ whose domain of applicability is decidable (in particular, for complete algorithms), the second regularity condition is always fulfilled.

Note: Figure translations are in progress. See original paper for figures.

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