

# ON CONDENSATIONS ONTO EUCLIDEAN SPACES

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**Abstract**

**Full Text**

**MATHEMATICS**

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## **ON CONDENSATIONS ONTO EUCLIDEAN SPACES**

*(Presented by Academician P. S. Aleksandrov, March 12, 1963)*

The main theorem of the present note is as follows:

**Theorem 1.** *Let a condensation  $f : X \rightarrow E^n$  be given, where  $X$  is a connected, locally bicomact paracompact space, and  $E^n$  is Euclidean space. Then  $f$  is a homeomorphism.*

For  $n = 1$  this theorem has a strengthening:

**Theorem 2.** *Let a condensation  $f : X \rightarrow L$  be given, where  $X$  is a connected and peripherally bicomact\*\* space, and  $L$  is a line. Then  $f$  is a homeomorphism.*

**Proof of Theorem 2.** In order that a condensation be a homeomorphism, it is necessary and sufficient that it be closed. Suppose that  $f$  is not closed. Then it is not hard to see that there will be a countable closed subset  $A \subset X$  such that  $fA \subset L$  is not closed. Let  $\xi = f^{-1}a$ , where  $a \in [fA] \setminus fA$ . The space  $X$  is Hausdorff, since it is condensed onto the Hausdorff space  $L$ , and, being peripherally bicomact, it is completely regular <sup>(1)</sup>. Therefore there exists at the point  $\xi$  a neighborhood  $O\xi$  with bicomact boundary  $F$  and such that  $[O\xi] \cap A = \Lambda$ , or, what is the same thing:  $f[O\xi] \cap fA = \Lambda$ . There is an  $\varepsilon > 0$  such that  $O_\varepsilon a \cap fF = \Lambda$ .

The line  $L$  is representable as the sum of two rays:  $L = L^+ \cup L^-$ ,  $L^+ \cap L^- = a$ . Essentially, two cases are logically possible:

1.  $a \in [(L^- \setminus a) \cap f[O\xi]]$ ,  $a \in [(L^+ \setminus a) \cap f[O\xi]]$ , and, in addition, for example,  $a \in [L^- \cap fA]$ . In this case we proceed as follows. Choose a point  $b$  from  $L^-$  such that  $b \in fO\xi$  and  $|a - b| < \varepsilon$ , where  $\varepsilon$  is the distance from the point  $a$  to  $fF$ . Denote the segment  $[a, b]$  by  $I$ . The set  $f^{-1}Q$ , where  $Q = I \setminus f[O\xi]$ , is open-and-closed in  $X$ . The set  $f^{-1}Q$  is open, since  $f^{-1}Q = f^{-1}(a, b) \setminus [O\xi]$ , and it is closed, for  $f^{-1}Q = f^{-1}I \setminus O\xi$ . But this contradicts the connectedness of  $X$ .
2.  $a \in [(L^- \setminus a) \cap f[O\xi]]$ ,  $a \notin [(L^+ \setminus a) \cap f[O\xi]]$ . In this case  $P = f^{-1}(L^+ \setminus a)$  is open-and-closed in  $X$ . It is obvious that  $P$  is open and that  $[P] \setminus P$  either consists of the single point  $f^{-1}a$ , or is empty. But at  $f^{-1}a$ , by virtue of

the conditions of this case, there is a neighborhood containing no points of the set  $P$ . Thus, the theorem is proved.

The following theorem also holds; I give it without proof.

**Theorem 3.** *Let a condensation  $f : X \rightarrow L$  be given, where  $X$  is a connected and locally connected space, and  $L$  is a line. Then  $f$  is a homeomorphism.*

**Proof of Theorem 1.** We shall call a point  $\xi \in E^n$  **bad** if the image of at least one neighborhood of the point  $f^{-1}\xi$  contains no neighborhood of the point  $\xi$ . It is not hard to verify that the set of all bad points  $W$  is closed in  $E^n$ . We shall show that  $W$  is nowhere dense in  $E^n$ . Suppose the contrary: an open ball  $Q$  consists entirely of bad points,  $Q \subset W$ . Let  $\xi \in Q$ ,  $x = f^{-1}\xi$ , and let a neighborhood  $Ox$  have bicomcompact closure  $[Ox] = B$ .

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\* A condensation is a one-to-one continuous mapping.

\*\* A space is peripherally bicomcompact if it has a base whose elements have bicomcompact boundaries.

If some point  $\eta \in fB$  were interior, then it could not be bad; hence  $fB$  is nowhere dense in  $Q$ . Since  $X$  is finally compact, it follows from what has been proved that  $Q$  can be represented as the sum of a countable number of closed nowhere dense sets, which contradicts the theorem. Thus  $W$  is closed and nowhere dense.

We shall call a spherical neighborhood  $O\xi$  of a point  $\xi \in W$  **special** if  $O\xi = O_1 \cup O_2$ , where  $f^{-1}O_1 = V$  is open-and-closed in  $f^{-1}O\xi$ , and  $V$  is a neighborhood of the point  $f^{-1}\xi$  whose closure is bicomcompact. Note that  $O_1 \cap |O_2| \subseteq W$ . The neighborhood  $V$  of the point  $f^{-1}\xi$  will be called **conjugate** to  $O\xi$ . Every point  $\xi \in W$  has special neighborhoods of arbitrarily small radius. Construct a neighborhood  $OW$  of the set  $W$ : mark an arbitrary  $\xi_0 \in W$  with its special neighborhood  $O\xi_0$ ; let the point  $\eta \in O\xi_0$ ,  $\eta \notin W$ , but  $f^{-1}\eta \in V_0$ , where  $V_0$  is conjugate to  $O\xi_0$ . The neighborhood  $OW$  is an arbitrary neighborhood of  $W$  that is the sum of special neighborhoods of points of  $W$ , each of which, except  $O\xi_0$ , does not contain  $\eta$ .

Since  $X \setminus f^{-1}W$  is mapped homeomorphically by  $f$  onto  $E^n \setminus W$ , the boundary of  $f^{-1}OW$  is mapped homeomorphically onto the boundary of  $OW$ . Mark the  $U$ -component of the set  $OW$  containing  $\xi_0$ . We note that  $f^{-1}U$  is disconnected,  $f^{-1}U = \Gamma_1 \cup \Gamma_2$ , where  $\Gamma_1$  is the sum of the neighborhoods  $V_\alpha$  conjugate to the special neighborhoods forming  $U$ , and  $\Gamma_2$  contains the point  $\eta$ . If  $R = W \cap U$  does not split  $U$ , then  $R$  does not split  $E^n$  either, and then  $f^{-1}R$  is open-and-closed in  $X$ , for  $U \setminus R$  is connected, while  $f^{-1}U$  is disconnected. But this contradicts the connectedness of  $X$ . Suppose now that  $R$  does split  $U$ ; then  $R$  splits  $E^n$ , since  $U$  was a connected neighborhood of the closed set  $R$ , and such a splitting can be carried out:

$$E^n \setminus R = \Delta_1 \cup \Delta_2,$$

where  $\Delta_1$  and  $\Delta_2$  are open in  $E^n \setminus R$ ,  $\Delta_1 \cap \Delta_2 = \Lambda$ , and also  $f\Gamma_1 \setminus R \subseteq \Delta_1$ ,  $f\Gamma_2 \subseteq \Delta_2$ . It is now not hard to see that  $f^{-1}\Delta_2$  is open-and-closed in  $X$ . This contradicts the connectedness of  $X$ , and therefore the set of bad points  $W$  is empty. The theorem is proved.

**Corollary 1.** Every condensation of  $E^n$  onto itself is a homeomorphism.

**Corollary 2.** Let a condensation be given  $f : X \rightarrow I^n$ , where  $X$  is connected, locally bicomact, and paracompact, and  $I^n$  is the  $n$ -dimensional cube. Then  $f$  is a homeomorphism.

**Corollary 3.** Let a condensation be given  $f : X \rightarrow S_n$  ( $n \neq 1$ ), where  $X$  is connected, locally bicomact, and paracompact, and  $S_n$  is the  $n$ -dimensional sphere. Then  $f$  is a homeomorphism.

Corollary 1 follows immediately from Theorem 1, but Corollaries 2 and 3 require proofs, which I omit.

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*Note added in proof.* Recently I obtained a result more general than Theorem 1:

**Theorem.** Let a condensation be given  $f : X \rightarrow Y$ , where  $X$  is a connected, locally bicomact space with a countable base, and  $Y$  is a locally connected, locally bicomact, and unicoherent space with a countable base. Then  $f$  is a homeomorphism.

## REFERENCES

1. E. G. Sklyarenko, DAN, **120**, No. 6, 1200 (1958).

*Note: Figure translations are in progress. See original paper for figures.*

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