Aerodynamics

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1963

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Abstract

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On the Nonlinear Theory of Aerodynamic Stability

To study the stability of laminar flows of a viscous liquid or gas, the equations of hydrodynamics are usually linearized with respect to disturbances that are assumed to be so small that their square may be neglected. In the linear theory of stability, a number of very interesting and practically important results have been obtained in the investigation of the stability boundaries of laminar flows and in the determination of the critical values of Reynolds numbers. However, as a result of the simplifications adopted, the conclusions of the linear theory concerning the character of the development of disturbances in time will be valid only for sufficiently small time intervals (stability in the small). According to the linear theory, for $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ small disturbances in the unstable region will grow in time according to an exponential law. In reality, however, when nonlinear terms in the equations of motion are taken into account, unstable flows may either continue to grow with time, destroying the original laminar flow and giving rise to turbulence, or become stabilized in time, leading to new, also laminar, flows. In this case the physical meaning and practical significance of the stability boundaries found by the linear theory will be entirely different.

In the present work, to investigate the problem of aerodynamic stability, the method of successive approximations—the method of the small parameter—is applied. The first approximation coincides with the usual linear theory. For the subsequent approximations a recurrent system of ordinary differential equations is obtained. It is shown in the paper that the stability boundaries and the character of the change of disturbances in time in the successive approximations are determined by the first—linear—approximation.

Let us consider the plane motion of a viscous incompressible fluid, whose stream function satisfies the equation

$$\frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \nabla^2 \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \nabla^2 \psi}{\partial y} = \gamma \nabla^2 \nabla^2 \psi. \tag{1}$$

Let the motion under consideration consist of a basic stationary motion with stream function $\psi_0(x, y)$ and a disturbed nonstationary motion with stream function $\psi_1(x,y,t)$. The stream function of the basic motion will satisfy equation (1), or the corresponding equation in boundary-layer theory. The stream function of the disturbed motion will satisfy the equation

$$\begin{aligned}\frac{\partial\nabla^2\psi_1}{\partial t}+U_0\frac{\partial\nabla^2\psi_1}{\partial x}+V_0\frac{\partial\nabla^2\psi_1}{\partial y}-\left(\nabla^2U_0\frac{\partial\psi_1}{\partial x}+\nabla^2V_0\frac{\partial\psi_1}{\partial y}\right)=\\ =\gamma\nabla^2\nabla^2\psi_1+\frac{\partial\psi_1}{\partial x}\frac{\partial\nabla^2\psi_1}{\partial y}-\frac{\partial\psi_1}{\partial y}\frac{\partial\nabla^2\psi_1}{\partial x}.\end{aligned}\tag{2}$$

In the motion of a fluid between two parallel walls (flow in a channel), the velocity component $V_0=0$, and the velocity component

$U_0=U_0(y)$. In this case equation (2) can be written in the form

$$\mathcal{L}(U_0\psi_1)=\frac{\partial\nabla^2\psi_1}{\partial t}+U_0\frac{\partial\nabla^2\psi_1}{\partial x}-U_0''\frac{\partial\psi_1}{\partial x}-\gamma\nabla^2\nabla^2\psi_1=\frac{\partial\psi_1}{\partial x}\frac{\partial\nabla^2\psi_1}{\partial y}-\frac{\partial\psi_1}{\partial y}\frac{\partial\nabla^2\psi_1}{\partial x}.\tag{3}$$

For the motion of a fluid in a boundary layer, the derivatives with respect to x of the stream function of the basic flow ψ_0 will be small compared with the derivatives with respect to y. In this case, in equation (2) the terms containing V_0 and Δ^2V_0 may be omitted, and it likewise reduces to equation (3).

Equation (3) differs from the corresponding equation considered in the linearized theory of stability by the right-hand side—by the nonlinear terms.

We shall seek a solution of equation (3) in the form

$$\psi_1(x,y,t)=\sum_{n=1}^{\infty}\varepsilon^n\varphi_n(x,y,t)\tag{4}$$

and assume that this series converges uniformly for sufficiently small values of ε and t.

Substituting (4) into (3) and collecting terms with equal powers of ε, we obtain the following recurrent system of differential equations:

$$\begin{gathered}\mathcal{L}(U_0\varphi_1)=0,\\ \mathcal{L}(U_0\varphi_2)=\frac{\partial\varphi_1}{\partial x}\frac{\partial\nabla^2\varphi_1}{\partial y}-\frac{\partial\varphi_1}{\partial y}\frac{\partial\nabla^2\varphi_1}{\partial x}=[\varphi_1\varphi_1],\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \mathcal{L}(U_0\varphi_m)=\sum_{n=1}^{m-1}\frac{\partial\varphi_n}{\partial x}\frac{\partial\nabla^2\varphi_{m-n}}{\partial y}-\frac{\partial\varphi_n}{\partial y}\frac{\partial\nabla^2\varphi_{m-n}}{\partial x}=\sum_{n=1}^{m-1}[\varphi_n\varphi_{m-n}].\end{gathered}\tag{5}$$

The first equation of this system coincides with the basic equation of the linear theory of stability. As is known, the solution of this equation is usually sought in the form

$$\Phi_1 = f_1(\alpha, \beta, y) e^{i(\alpha x - \beta t)}. \tag{6}$$

Then the function $f_1(\alpha, \beta, y)$ will satisfy the ordinary differential equation—the Orr–Sommerfeld equation:

$$Q(\alpha, \beta, f_1) = i\alpha \left[\left(U_0 - \frac{\beta}{\alpha} \right) \{f_1'' - \alpha^2 f_1\} - U_0'' f_1 \right] - \gamma \{f_1^{(\mathrm{IV})} - 2\alpha^2 f_1'' + \alpha^4 f_1\} = 0. \tag{7}$$

The function ψ_1, and consequently also the function $f_1(y)$, must satisfy certain boundary conditions.

For the motion of a fluid between two parallel walls separated by a distance h:

$$f_1(0) = f_1'(0) = f_1(h) = f_1'(h) = 0. \tag{8}$$

For the motion of a fluid in a boundary layer: on the surface of the body being flowed around, $f_1(0) = f_1'(0) = 0$, and the conditions are fulfilled for the transformation of the solution at the boundary of the boundary layer into the solution for an ideal fluid.

In order that the solution (6) satisfy the boundary conditions (8), it is necessary that the parameter β be a quite definite function of the parameter α and of the Reynolds number:

$$\beta = \omega(\alpha \,\mathrm{Re}) + i\gamma_1(\alpha \,\mathrm{Re}).$$

Consequently, the solution (6) may be written in the form

$$\Phi_1 = \varphi_1 + i\psi_1 = f_1(\alpha, y) e^{i(\alpha x - \omega t) + \gamma_1 t}. \tag{9}$$

In the region of stable flows, $\gamma_1(\alpha \,\mathrm{Re}) < 0$; in the region of unstable flows, $\gamma_1(\alpha \,\mathrm{Re}) > 0$.

On the basis of the foregoing, a particular solution of the recurrent system of differential equations (5) may be sought in the form

$$\varphi_n = F_n(x, y, t) e^{\gamma_1 n t}, \tag{10}$$

where $F_n(x, y, t)$ are periodic functions of the variables x and t. Substituting (10) into (5), we obtain the following recurrent system of differential equations for determining the periodic functions $F_n(x, y, t)$:

$$\mathcal{L}(U_0 F_2) + 2\gamma_1 \nabla^2 F_2 = \frac{\partial F_1}{\partial x}\frac{\partial \nabla^2 F_1}{\partial y} - \frac{\partial F_1}{\partial y}\frac{\partial \nabla^2 F_1}{\partial x},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathcal{L}(U_0 F_m) + m\gamma_1 \nabla^2 F_m = \sum_{n=1}^{m-1} \frac{\partial F_n}{\partial x}\frac{\partial \nabla^2 F_{m-n}}{\partial y} - \frac{\partial F_n}{\partial y}\frac{\partial F_{m-n}}{\partial x}.$$

Consequently, a particular solution for the stream functions of the perturbed motion may be represented in the form

$$\psi_1(x, y, t) = \sum_{n=1}^{\infty} \varepsilon^n F_n(x, y, t) e^{\gamma_1 n t}, \tag{11}$$

where ε is a small parameter, and $\gamma_1(\alpha, \mathrm{Re})$ is determined from the solution of the linearized problem.

As is seen, in the region of stable values $\gamma_1(\alpha\,\mathrm{Re}) < 0$, the first and subsequent approximations will tend to zero. In the region of unstable flows, $\gamma_1(\alpha\,\mathrm{Re}) > 0$, the first and subsequent approximations will grow with time.

Let us show that, in order to determine the sequence of unknown functions F_n, one can obtain a recurrent system of ordinary differential equations and reduce the problem to quadratures.

We shall start from the system of differential equations (5) and, instead of the real functions $\varphi_1, \varphi_2, \ldots, \varphi_m$, just as is customary in the linearized problem, introduce the following system of complex functions:

$$\Phi_1 = \varphi_1 + i\psi_1$$

$$\Phi_2 = \varphi_2 + i\psi_2 \tag{12}$$

$$\ldots\ldots\ldots$$

$$\Phi_m = \varphi_m + i\psi_m.$$

Since the system of functions $\varphi_1, \dots, \varphi_m$ satisfies the system of equations (5), the system of complex functions (12) will satisfy a certain modified system of equations.

Using the identity of the form

$$\varphi^2 + i\varphi\psi = \frac{1}{2}\Phi^2 + \frac{1}{2}\Phi\widetilde{\Phi},$$

we write this modified system of equations in the form

$$\mathcal{L}(U_0\Phi_1) = 0$$

$$\mathcal{L}(U_0\Phi_2) = \frac{1}{2}\left[\frac{\partial\Phi_1}{\partial x}\frac{\partial\nabla^2\Phi_1}{\partial y} - \frac{\partial\Phi_1}{\partial y}\frac{\partial\nabla^2\Phi_1}{\partial x}\right] + \frac{1}{2}[\Phi_1\widetilde{\Phi}_1],$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\mathcal{L}(U_0\Phi_m) = \frac{1}{2}\sum_{n=1}^{m-1}\left\{\left[\frac{\partial\Phi_n}{\partial x}\frac{\partial\nabla^2\Phi_{m-n}}{\partial y} - \frac{\partial\Phi_n}{\partial y}\frac{\partial\nabla^2\Phi_{m-n}}{\partial x}\right] + [\Phi_n\widetilde{\Phi}_{m-n}]\right\}.$$

The first equation of this system reduces to the Orr–Sommerfeld equation. The second equation, after substitution of the expression (9) into the right-hand side, will have the form

$$\mathcal{L}(U_0\Phi_2) = \frac{i\alpha}{2}\{f_1 f_1''' - f_1'' f_1'\}e^{2i(\alpha x - \omega t) + 2\gamma_1 t} - \frac{i\alpha}{2}\{\tilde{f}_1(f_1' - \alpha^2 f_1)\}' e^{2\gamma_1 t}.$$

Consequently, the function Φ_2 can be represented in the form

$$\Phi_2 = f_2(2\alpha, 2\beta, y)e^{i2(\alpha x - \beta t)} + f_0(2\gamma_1, y)e^{2\gamma_1 t}. \tag{13}$$

Then $f_2(2\alpha, 2\beta, y)$ must satisfy the nonhomogeneous Orr—Sommerfeld differential equation

$$Q(2\alpha, 2\beta, f_2) = \frac{i\alpha}{2}\left(f_1 f_1''' - f_1'' f_1'\right). \tag{14}$$

The function $f_0(y, 2\gamma_1)$ must satisfy the following nonhomogeneous differential equation:

$$2\gamma_1 \frac{d^2 f_0}{dy^2} - \gamma \frac{d^4 f_0}{dy^4} = -\frac{i\alpha}{2} \left\{ \tilde{f}_1 \left(f_1'' - \alpha^2 f_1 \right) \right\}' . \tag{15}$$

The solution of equations (14) and (15) can be written in quadratures. Analogously, solutions can be written for the functions Φ_m of higher order. As is clear from what has been said, the real parts of the functions Φ_m will satisfy the recurrent system of differential equations (5) and, consequently, the original nonlinear equation (3). A particular solution of this nonlinear differential equation for the perturbed motion can also be written in the form

$$\psi_1 = \sum_{n=1}^{\infty} \varepsilon^n F_n(x, y, t) e^{n\gamma_1 t}, \tag{16}$$

where

$$\begin{aligned} F_1(xyt) &= a_1(y)\cos(\alpha x - \omega t) + b_1(y)\sin(\alpha x - \omega t), \\ F_2(xyt) &= a_2(y)\cos 2(\alpha x - \omega t) + b_2(y)\sin 2(\alpha x - \omega t) + b_0(y), \\ F_3(xyt) &= a_3(y)\cos 3(\alpha x - \omega t) + b_3(y)\sin 3(\alpha x - \omega t) + c_1(y)\cos(\alpha x - \omega t) + \\ &\quad + d_1(y)\sin(\alpha x - \omega t). \end{aligned}$$

Consequently, in the nonlinear theory of aerodynamic stability, the stability boundaries and the character of the change of disturbances with time in successive approximations are determined by the parameter $\gamma_1(\alpha\,\mathrm{Re})$, borrowed from the linear theory. As is seen, for negative values of the parameter γ_1, the first and subsequent approximations will decrease with time. Consequently, laminar flows that are stable according to the linear theory (stable in the small) will also be stable according to the nonlinear theory (stable as a whole).

For $\gamma_1(\alpha\,\mathrm{Re}) = 0$, from (16) we formally obtain solutions periodic in time. However, this case requires further, more detailed study.

For positive values of $\gamma_1(\alpha\,\mathrm{Re})$ in the unstable region of the flow, the first and subsequent approximations will grow with time. For small disturbances, in the first period of time only the first terms of expansions (11) and (16) will be significant. However, in the subsequent period of time, beyond the limits of applicability of the linear theory, the subsequent terms of the expansions will also be significant. As is seen, in the nonlinear theory small disturbances will grow with time according to a considerably more complicated law than in the linear theory. One may now expect a slower development of disturbances with time and even their stabilization. However, in order to investigate the character of the development in time of small disturbances and to determine the behavior of solutions (11) and (16) as $t \to \infty$, it is necessary to estimate the convergence of the process of successive approximations, which, apparently, is difficult to do in the general case and should be considered for separate concrete examples of flows.

The author expresses gratitude to Academician A. A. Dorodnitsyn for valuable comments made by him in connection with this work.

Received
26 XI 1962

Note: Figure translations are in progress. See original paper for figures.

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