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Mathematics

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Abstract

Full Text

Mathematics

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On Periodic Solutions of Differential-Operator Equations with a Small Parameter at the Derivative

(Presented by Academician I. G. Petrovskii, 14 VI 1962)

1. Consider a Banach space E, complex or real. By $\widetilde{E}(\omega)$ we shall denote the Banach space of ω-periodic continuous functions $\tilde{x} = x(t)$ with values in E; the norm in $\widetilde{E}(\omega)$ is defined by the equality

   $$\|\tilde{x}\| = \max \|x(t)\|, \quad 0 \leqslant t \leqslant \omega.$$

   By R we denote the space of linear bounded operations A in E; then $\widetilde{R}(\omega)$ consists of operator-functions $\widetilde{A} = A(t)$, continuously (in norm) depending on t and ω-periodic, with

   $$\|\widetilde{A}\| = \max \|A(t)\|, \quad 0 \leqslant t \leqslant \omega.$$

   Let E_1 and E_2 be two Banach spaces; denote the direct sum of the spaces $\widetilde{E}_1(\omega)$ and $\widetilde{E}_2(\omega)$ by $\widetilde{E}_{12}(\omega)$. Let $A(t)$ and $B(t)$ be ω-periodic operators in E_1 and E_2, depending on the parameters $\tilde{x}$ and $\tilde{y}$; we shall write $A(t) = A[\tilde{x}, \tilde{y}](t)$, $B(t) = B[\tilde{x}, \tilde{y}](t)$, where $A[\tilde{x}, \tilde{y}]$ and $B[\tilde{x}, \tilde{y}]$ denote mappings, generally speaking nonlinear, of the space $\widetilde{E}_{12}(\omega)$ into $\widetilde{R}_1(\omega)$ and $\widetilde{R}_2(\omega)$. Let $f(\varepsilon, \tilde{x}, \tilde{y})$ and $g(\varepsilon, \tilde{x}, \tilde{y})$ be mappings of the space $\widetilde{E}_{12}(\omega)$ into $\widetilde{E}_1(\omega)$ and $\widetilde{E}_2(\omega)$, depending on the small parameter ε.

The following differential-operator equations make sense:

$$\varepsilon \frac{dx}{dt} + A[\tilde{x}, \tilde{y}](t)x = f(\varepsilon, \tilde{x}, \tilde{y})(t),$$

$$\frac{dy}{dt} + B[\tilde{x}, \tilde{y}](t)y = g(\varepsilon, \tilde{x}, \tilde{y})(t), \tag{1}$$

where $x(t)$ and $y(t)$ are unknown functions of period ω with values in E_1 and E_2.

In the case of finite-dimensional spaces E_1 and E_2, system (1) contains, in particular, the system of ordinary differential equations studied in (^{1}), as well as the system of integro-differential equations considered in (^{2}). In the case of infinite-dimensional spaces, system (1) can be realized as integro-differential equations in partial derivatives.

We shall also consider the case when $A[\tilde{x},\tilde{y}] = A(t)$ does not depend on $\tilde{x},\tilde{y}$ and is an unbounded operator generated, for example, by a boundary-value problem for a parabolic equation.

For system (1) the existence will be proved, for small ε, of an ω-periodic solution tending as $\varepsilon \to 0$ to a smooth ω-periodic solution of the degenerate system ($\varepsilon = 0$), which is assumed to be known. To this end we reduce problem (1), by generalizing the method of (^{1}), to a nonlinear integral equation, to which we apply the Schauder principle and the principle of contraction mappings.

2. We consider system (1) as a system of the form

$$x' + \varepsilon^{-1}A(t)x = \varepsilon^{-1}f(t), \qquad y' + B(t)y = g(t)$$

with periodic $A(t), B(t), f(t), g(t)$. It is easy to express the unique···

under certain conditions, the ω-periodic solution of this system by means of the fundamental-solution operators $U_\varepsilon(t,s)$, $U(t,s)$ ($U(s,s) = I$) in terms of $f(t)$ and $g(t)$. This will lead us to the integral equations

$$x(t) = \varepsilon^{-1}U_\varepsilon(t,0;\tilde{x},\tilde{y})[I - U_\varepsilon(\omega,0;\tilde{x},\tilde{y})]^{-1}\int_0^\omega U_\varepsilon(\omega,s;\tilde{x},\tilde{y})f(\varepsilon,\tilde{x},\tilde{y})(s)\,ds +$$

$$+\varepsilon^{-1}\int_0^t U_\varepsilon(t,s;\tilde{x},\tilde{y})f(\varepsilon,\tilde{x},\tilde{y})(s)\,ds, \tag{2}$$

$$y(t) = U(t,0;\tilde{x},\tilde{y})[I - U(\omega,0;\tilde{x},\tilde{y})]^{-1}\int_0^\omega U(\omega,s;\tilde{x},\tilde{y})g(\varepsilon,\tilde{x},\tilde{y})(s)\,ds +$$

$$+\int_0^t U(t,s;\tilde{x},\tilde{y})g(\varepsilon,\tilde{x},\tilde{y})(s)\,ds, \tag{3}$$

considered in the space $\widetilde{E}_{12}(\omega)$ of abstract functions $x(t), y(t)$ defined on a circle of length ω. Problem (1) is equivalent to equations (2)–(3). The operator generated by the right-hand side of (2)–(3) will be denoted by $\{F_1(\varepsilon,\tilde{x},\tilde{y}), F_2(\varepsilon,\tilde{x},\tilde{y})\}$; it acts in $\widetilde{E}_{12}(\omega)$.

3. Consider the equation

$$x' + \varepsilon^{-1}A(t)x = 0 \tag{4}$$

with continuous operator $A(t)$. Suppose that the fundamental solution $U_\varepsilon(t,s;A)$ for each fixed $\varepsilon \neq 0$ satisfies the estimate

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$$\|U_\varepsilon(t,s;A)\| \le N e^{-\frac{\delta}{|\varepsilon|}(t-s)}, \qquad t \ge s \ge 0,$$

or

$$\|U_\varepsilon(t,s;A)\| \le N e^{-\frac{\delta}{|\varepsilon|}(s-t)}, \qquad s \ge t \ge 0, \tag{5}$$

where $\delta > 0$ does not depend on ε.

We indicate some criteria for condition (5) to hold.

Lemma 1. If for any $t \ge s \ge 0$ the estimate

$$\left\|e^{-(t-s)A(t)}\right\| \le e^{-\delta(t-s)}, \qquad \delta > 0, \tag{6}$$

is valid, then for $\varepsilon > 0$ the first estimate (5) is true, and for $\varepsilon < 0$ the second.

If $A(t) = \text{const} = A$, then condition (5) is equivalent to the condition of uniform asymptotic stability of the zero solution of the equation $x' + Ax = 0$ as $t \to +\infty$ or $t \to -\infty$.

If the resolvent set of the operator $A(t)$ contains all real numbers $\alpha \le 1$ and $\|[I - \alpha A]^{-1}\| \le (1-\alpha)^{-1}$, then (6) is valid$^{(4)}$. In the case of a Hilbert space, condition (6) follows from the inequality $\operatorname{Re}(A(t)x, x) \ge \delta(x,x)$, $\delta > 0$, $x \in H$.

Let us note that condition (5) ensures that the operator $\{F_1, F_2\}$ acts in some ball, if ε is small.

4. We now formulate a theorem on the solvability of equation (1). Suppose that the degenerate system ($\varepsilon = 0$) has a smooth ω-periodic solution $\tilde{x}_0, \tilde{y}_0$ and that:

1) there exists a continuously differentiable ω-periodic and invertible operator $P(t)$ reducing the operator $A[\tilde{x}_0, \tilde{y}_0](t)$, i.e. the space E_1 decomposes into the direct sum $E_1 = E_1^- + E_1^+$ and

$$P^{-1}(t)A[\tilde{x}_0, \tilde{y}_0](t)P(t) = \begin{pmatrix} A_-(t) & 0 \\ 0 & A_+(t) \end{pmatrix},$$

where A_- acts in E_1^- and satisfies the first condition (5), while A_+ satisfies the second and acts in E_1^+;

2) the operator $B[\tilde{x}_0, \tilde{y}_0](t)$ generates the monodromy operator $U(\omega, 0)$, which has 1 as a regular value;

3) the operators f and g consist of a finite number of terms f_ν and g_ν, satisfying one of the conditions $3_1), 3_2), 3_3)$:

3_1)

$$f_\nu(\varepsilon, \tilde{x}, \tilde{y}) - f_\nu(0, \tilde{x}_0, \tilde{y}_0) \to 0$$

as $\varepsilon \to 0$, uniformly with respect to $\tilde{x}, \tilde{y}$ from some neighborhood of the point $\tilde{x}_0, \tilde{y}_0$; similarly for g_ν;

3_2) for f_ν the estimate

$$\|f_\nu(\varepsilon, \tilde{x}, \tilde{y}) - f_\nu(\varepsilon, \tilde{x}_0, \tilde{y}_0)\| \leq K(\varepsilon, \tilde{x}, \tilde{y})\|\tilde{x} - \tilde{x}_0\| + \|\tilde{y} - \tilde{y}_0\|,$$

holds, where $K \to 0$ as $\varepsilon \to 0,\ \tilde{x} \to \tilde{x}_0,\ \tilde{y} \to \tilde{y}_0$, and $f_\nu(\varepsilon, \tilde{x}_0, \tilde{y}_0)$ is continuous in ε at $\varepsilon = 0$; similarly for g_ν;

3_3) in a neighborhood of the point $\tilde{x}_0, \tilde{y}_0$, g_ν satisfies the inequality

$$\|g_\nu(\varepsilon, \tilde{x}, \tilde{y}) - g_\nu(0, \tilde{x}_0, \tilde{y}_0)\| \leq M\|\tilde{x} - \tilde{x}_0\|, \qquad M = \text{const}.$$

Theorem 1. *Suppose that, in addition to* $1) - 3)$*, the following conditions are satisfied:*

4) *the operators* $A[\tilde{x}, \tilde{y}]$*,* $B[\tilde{x}, \tilde{y}]$ *are completely continuous in a neighborhood of the point* $\tilde{x}_0, \tilde{y}_0$*, and the Fréchet derivative at this point of the operators* $A[\tilde{x}, \tilde{y}]\tilde{x}_0$*,* $B[\tilde{x}, \tilde{y}]\tilde{y}_0$ *is equal to zero;*

5) $P'(t)$ *is completely continuous for each* t*;*

6) *the operators* f *and* g *are completely continuous in a neighborhood of* $\tilde{x}_0, \tilde{y}_0$ *for small* ε*.*

Then, as $\varepsilon \to 0$*, system (1) has an* ω*-periodic solution* $x_\varepsilon(t), y_\varepsilon(t)$*, which tends uniformly to the solution* $\tilde{x}_0, \tilde{y}_0$ *of the degenerate system.*

Let us formulate one more result:

Suppose that conditions $1) - 3)$ *are satisfied and, in addition:*

$4'$) $A[\tilde{x}, \tilde{y}]$*,* $B[\tilde{x}, \tilde{y}]$ *in a neighborhood of the point* $\tilde{x}_0, \tilde{y}_0$ *satisfy the Lipschitz condition;*

$5'$) $A[\tilde{x}, \tilde{y}]\tilde{x}_0$*,* $B[\tilde{x}, \tilde{y}]\tilde{y}_0$ *satisfy the Lipschitz condition with an arbitrarily small constant as* $\tilde{x} \to \tilde{x}_0,\ \tilde{y} \to \tilde{y}_0$*;*

$6'$) *the operators* f_ν, g_ν *are either completely continuous or satisfy the Lipschitz condition with constant* $K^\nu_{\varepsilon,r}$ *in the neighborhood*

$$\|\tilde{x} - \tilde{x}_0\| \leq r, \qquad \|\tilde{y} - \tilde{y}_0\| \leq r,$$

where $K^\nu_{\varepsilon,r} \to 0$ *together with* ε *and* r*.*

Then the assertion of Theorem 1 is valid.

We note that a more general system with a full "linear part" can be reduced to form (1) by the same method as in paper (^{1}).

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5. Let us extend the results of Section 4 to the case where the operator $A[\tilde{x}, \tilde{y}]$ does not depend on $\tilde{x}, \tilde{y}$ and is an unbounded operator $A(t)$ with domain of definition $D(A)$, everywhere dense in E_1 and independent of t. Using the results of papers (4–6), one can construct integral equations (2)–(3) with estimate (5). We note that an equation of type (3) in the case of an unbounded operator B was mentioned in another connection at a seminar of M. A. Krasnosel' skii.

We impose the following conditions:

a) the operator $-A(t)$ is ω-periodic and generates, for a strongly continuous semigroup $e^{-\tau A(t)}$, $\tau \geq 0$, with

$$\|e^{-\tau A(t)}\| \leq e^{-\tau};$$

b) $A(t)A^{-1}(0)$ is strongly continuously differentiable with respect to t;

c) the function $f(\varepsilon, \tilde{x}, \tilde{y})(t)$ is strongly continuously differentiable with respect to t for each pair $\tilde{x}, \tilde{y}$;

d) the operator $B[\tilde{x}, \tilde{y}]$ is completely continuous in a neighborhood of the point $\tilde{x}_0, \tilde{y}_0$, and the Fréchet derivative of the operator $B[\tilde{x}, \tilde{y}]\tilde{y}_0$ at this point is equal to zero;

Γ*) $B[\tilde{x}, \tilde{y}]$ in a neighborhood of $\tilde{x}_0, \tilde{y}_0$ satisfies the Lipschitz condition, and $B[\tilde{x}, \tilde{y}]\tilde{y}_0$ satisfies the Lipschitz condition with an arbitrarily small constant as $\tilde{x} \to \tilde{x}_0$, $\tilde{y} \to \tilde{y}_0$.

Theorem 2. Suppose that conditions 2) and 3) of Theorem 1, conditions a)–) or a)– *) are satisfied, and, in addition:

) the operators f_v, g_v are either all completely continuous (under a)–)), or some of them, and hence all, satisfy the Lipschitz condition (under a)– *) in the neighborhood $\|\tilde{x} - \tilde{x}_0\| \leq r$, $\|\tilde{y} - \tilde{y}_0\| \leq r$, with constant $K_{\varepsilon,r} \to 0$ together with ε, r.

Then, as $\varepsilon \to 0$, there exists an ω-periodic solution of system (1), converging uniformly to the solution $\tilde{x}_0, \tilde{y}_0$ of the degenerate system.

Introducing the notion of a fractional power (5,6) of the operator A, one can show that the operators (2)–(3) are sometimes completely continuous even without condition) of Theorem 2. Let us formulate the corresponding assertion:

Theorem 3. Suppose that conditions 2), 3) of Theorem 1, conditions a)–) are satisfied, and

′) f is a continuous operator in a neighborhood of $\tilde{x}_0, \tilde{y}_0$; g, $A^{-1}(0)$ are completely continuous;

′) $\|[A(t) + \lambda I]^{-1}\| \leq [1 + |\lambda|]^{-1}$, $\operatorname{Re}\lambda \geq 0$.

Then, as $\varepsilon \to 0$, system (1) has an ω-periodic solution converging uniformly to the solution of the degenerate ($\varepsilon = 0$) system.

Let us formulate one more assertion:

Theorem 4. Suppose that the operator A does not depend on t and, together with the operators $B[A^{-\alpha}\tilde{x}, \tilde{y}]$, $f(\varepsilon, A^{-\alpha}\tilde{x}, \tilde{y})$, $g(\varepsilon, A^{-\alpha}\tilde{x}, \tilde{y})$, satisfies the conditions of the preceding theorem for some $0 \le \alpha < 1$, and with $A^{\alpha}\tilde{x}_0, \tilde{y}_0$ in place of $\tilde{x}_0, \tilde{y}_0$.

Then the assertion of Theorem 3 is valid.

We now consider the weakly coupled system of equations

$$\varepsilon \frac{dx}{dt} + A(t)x = f(\varepsilon, \tilde{x}, \tilde{y}),$$

$$\frac{dy}{dt} + B(t)y = g(\varepsilon, \tilde{x}, \tilde{y}) \tag{7}$$

with unbounded operators $A(t)$ and $B(t)$. Suppose that the degenerate system ($\varepsilon = 0$) has a smooth ω-periodic solution $\tilde{x}_0, \tilde{y}_0$.

We introduce the corresponding changes into the formulations of Theorems 2—4: the condition on the operator $B[\tilde{x}, \tilde{y}]$ is dropped, while g and $B(t)$ are assumed to satisfy the same conditions as f and $A(t)$, the conditions for which remain unchanged; if A and B do not depend on t, then the preceding conditions need only be imposed on $f(\varepsilon, A^{-\alpha}\tilde{x}, B^{-\beta}\tilde{y})$ and $g(\varepsilon, A^{-\alpha}\tilde{x}, B^{-\beta}\tilde{y})$ for some $0 \le \alpha, \beta < 1$, and with $A^{\alpha}x_0$, $B^{\beta}y_0$ in place of x_0, y_0. Then the assertions of the preceding theorems are valid.

We note that the last theorems make it possible to consider unbounded nonlinear operators B, f, g.

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