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Abstract

Full Text

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ON THE DISTRIBUTIONS OF POLYNOMIAL STATISTICS IN SAMPLES FROM A NORMAL POPULATION

(Presented by Academician V. I. Smirnov on 10 X 1962)

Let $x_1, x_2, \dots, x_k$ be a repeated homogeneous sample of size $k \geqslant 1$ from a normal population with mean 0 and variance 1. A known property (see (1)) of the distribution of a quadratic form in the variables $x_1, x_2, \dots, x_k$ is that the characteristic function $\varphi(t)$ of this distribution has the form

$$\varphi(t) = \prod_{\nu=1}^{k}(1-2i\lambda_\nu t)^{-1/2}, \tag{1}$$

where $\lambda_1, \lambda_2, \dots, \lambda_k$ are the characteristic roots of the quadratic form. From (1) it follows directly that $\varphi(t)$ satisfies a linear homogeneous differential equation with polynomial coefficients of the form

$$P(t)\varphi' + {}^1/_2 P'(t)\varphi = 0, \tag{2}$$

where $P(t) = \prod_{\nu=1}^{k}(1-2i\lambda_\nu t)$. This property of the distributions of quadratic forms in normal variables can be generalized to the case of polynomial statistics with real coefficients of higher degrees.

We shall call a form in the variables $x_1, x_2, \dots, x_k$ *suitable* if its partial derivatives with respect to $x_1, x_2, \dots, x_k$ have no common root, except perhaps the trivial one $(0, 0, \dots, 0)$. Examples of suitable forms are any definite quadratic form, $x_1^n + x_2^n + \dots + x_k^n$, and the like. We shall call a polynomial statistic suitable if the form of highest degree contained in it is suitable.

Let $X = (X_1, X_2, \dots, X_k)$ be a random vector having a nondegenerate normal distribution, and let $P(X) = P(X_1, X_2, \dots, X_k)$ be a polynomial statistic with real coefficients. For an arbitrary polynomial statistic $F(X)$, define the function of τ

$$\varphi(\tau, F) = E\left[F(X)\exp(\tau P(X))\right], \qquad \tau = it\ (-\infty < t < +\infty). \tag{3}$$

For arbitrary positive integers $N < M$, we shall consider the set $\Phi = \{\varphi_{j_1, j_2, \dots, j_k}(\tau)\}$, where

$$\varphi_{j_1,j_2,\dots,j_k}(\tau) = E\,[X_1^{j_1} X_2^{j_2} \cdots X_k^{j_k} \exp(\tau P(X))], \qquad \tau = it\ (-\infty < t < +\infty), \tag{4}$$

and $(j_1, j_2, \dots, j_k)$ runs through all possible sets of nonnegative integers satisfying the relation

$$N \leqslant j_1 + j_2 + \cdots + j_k \leqslant M.$$

Denote by $\mathcal{L}_{NM}(\tau)$ the vector whose components are the elements of the set Φ, arranged in some fixed order.

Theorem 1. If $P(X)$ is an appropriate polynomial statistic, then for any polynomial statistic $F(X)$ the function $\varphi(\tau, F)$ is a solution of some linear homogeneous differential equation with polynomial coefficients.

Theorem 2. If $P(X)$ is an appropriate form of degree $n+1 > 3$, $EX = 0$, then one can specify such a positive integer N_0 that the vector $\mathcal{L}_{N_0,N_0+n}(\tau)$ satisfies the differential equation

$$\frac{d\mathcal{L}}{d\tau} = \Omega(\tau)\mathcal{L}, \tag{5}$$

$$\tau = it\ (-\infty < t < +\infty),$$

where $\Omega = A/\tau + B/\tau^2$. Here A, B are constant square matrices of the corresponding order, and the elements of the matrix A depend only on the degree of the form P, but do not depend on its coefficients.

Remark. For $n+1 = 3$ the theorem remains valid, only in this case Ω has the form

$$\Omega = \frac{A}{\tau} + \frac{B}{\tau^2} + \frac{C}{\tau^3}.$$

From Theorem 1 there follows directly the generalization mentioned at the beginning. Namely:

Theorem 3. The characteristic function of an appropriate polynomial statistic satisfies a linear homogeneous differential equation with polynomial coefficients.

Application of the Fourier transform makes it possible to obtain a similar equation also for the distribution function of an appropriate polynomial statistic. This equation can be used to study certain probabilistic properties of these statistics.

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CITED LITERATURE

[1] H. Cramér, *Mathematical Methods of Statistics*, IL, 1948.

Note: Figure translations are in progress. See original paper for figures.