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Abstract

Full Text

THEORY OF ELASTICITY

M. A. MALKOV

A TWO-DIMENSIONAL PROBLEM ON THE ELASTIC COLLISION OF RODS

(Presented by Academician S. L. Sobolev on 13 VIII 1962)

Since the solution of problems on the collision of cylindrical rods reduces to quadratures from the solution of the two-dimensional problem on the collision of flat rods ([1]), we proceed to finding the latter.

Let two flat semi-infinite rods move toward one another with the same velocity v; the collision occurs at the time $t = 0$, and the coordinate system is chosen as shown in Fig. 1.

The interaction of the rods is described by the equations of motion of elastic bodies

$$\frac{\partial^2 u_x}{\partial t^2} = a^2 \frac{\partial \Delta}{\partial x} - b^2 \frac{\partial \omega}{\partial y}, \qquad \frac{\partial^2 u_y}{\partial t^2} = a^2 \frac{\partial \Delta}{\partial y} + b^2 \frac{\partial \omega}{\partial x}, \tag{1}$$

where a is the velocity of propagation of longitudinal disturbances, b that of transverse disturbances ($a \geq \sqrt{2}\, b$); u_x and u_y are the components of the displacement vector u; Δ is the volume-strain function: $\Delta = \partial u_x / \partial x + \partial u_y / \partial y$; ω is the shear-strain function: $\omega = \partial u_y / \partial x - \partial u_x / \partial y$.

According to equations (1), the functions Δ and ω satisfy the wave equations

$$\frac{\partial^2 \Delta}{\partial t^2} = a^2 \left(\frac{\partial^2 \Delta}{\partial x^2} + \frac{\partial^2 \Delta}{\partial y^2} \right), \qquad \frac{\partial^2 \omega}{\partial t^2} = b^2 \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right).$$

Fig. 1. Origin and propagation of rupture lines during the collision of rods

The solution of equations (1) must satisfy the initial conditions at $t = 0$:

$$u_x = 0, \qquad u_y = 0, \qquad \frac{\partial u_x}{\partial t} = 0, \qquad \frac{\partial u_y}{\partial t} = -v \text{ for } y > 0, \qquad \frac{\partial u_y}{\partial t} = v \text{ for } y < 0$$

and the boundary conditions at $|x| = d/2$, where d is the thickness of the rods:

$$a^2 \frac{\partial u_x}{\partial x} + (a^2 - 2b^2) \frac{\partial u_y}{\partial y} = 0,$$

$$\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = 0.$$

There exist several methods for solving problems of elastic impact, for example the methods of Hertz, Fourier, Laplace, Sobolev, and others. It can be shown that the Hertz method, for practically all materials, overestimates the average pressu-

at the contact point of the colliding bodies, i.e., it overestimates the resistance force during impact. A critique of the Fourier method was given by S. L. Sobolev. The critique of the Laplace method is analogous. S. L. Sobolev' s method is the most perfect; however, as a rule, this method is applied to potentials or displacement functions, which makes it somewhat cumbersome. In the present work this method is applied to the strain functions Δ and ω.

After the collision of the rods, the discontinuity in the initial conditions decomposes, leading to the appearance of a number of moving lines of discontinuity (Fig. 1).

The solution in region I has the form

$$\Delta = \omega = 0, \qquad u_x = 0,$$

$$u_y = -vt \quad \text{for } y > 0,$$

$$u_y = vt \quad \text{for } y < 0.$$

In region II

$$\Delta = -\frac{v}{a}, \qquad \omega = 0,$$

$$u_x = 0, \qquad u_y = -\frac{v}{a} y.$$

Fig. 2. Volumetric strains along the lateral surface of the rods. 1—$\beta = 0$; 2—$\beta^2 = 0.1$; 3—$\beta^2 = 0.2$; 4—$\beta^2 = 0.3$; 5—$\beta^2 = 0.4$; 6—$\beta^2 = 0.5$; a—two-dimensional solution; b—one-dimensional approximation. $v_0 = v/a$.

Figure 2: Fig. 2. Volumetric strains along the lateral surface of the rods. 1 —$\beta = 0$; 2—$\beta^2 = 0.1$; 3—$\beta^2 = 0.2$; 4—$\beta^2 = 0.3$; 5—$\beta^2 = 0.4$; 6—$\beta^2 = 0.5$; a—two-dimensional solution; b—one-dimensional approximation. $v_0 = v/a$.

Fig. 2. Volumetric strains along the lateral surface of the rods.
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The solution in region III is more conveniently sought in a polar coordinate system (ρ, φ), whose pole is placed at the point with coordinates $|x| = d/2$, $y = 0$.

By virtue of the self-similarity of the solution:

$$\Delta = \Delta\left(\frac{\rho}{at},\ \varphi,\ \frac{v}{a},\ \frac{b}{a}\right),$$

$$\omega = \omega\left(\frac{\rho}{at},\ \varphi,\ \frac{v}{a},\ \frac{b}{a}\right),$$

$$u_\rho = at\,U_\rho\left(\frac{\rho}{at},\ \varphi,\ \frac{v}{a},\ \frac{b}{a}\right),$$

$$u_\varphi = at\,U_\varphi\left(\frac{\rho}{at},\ \varphi,\ \frac{v}{a},\ \frac{b}{a}\right)$$

the basic equations (1) take the form:

$$r^2\frac{\partial^2 U_\rho}{\partial r^2} = \frac{\partial \Delta}{\partial r} - \frac{\beta^2}{r}\frac{\partial \omega}{\partial \varphi},$$

$$r^2\frac{\partial^2 U_\varphi}{\partial r^2} = \frac{1}{r}\frac{\partial \Delta}{\partial \varphi} + \beta^2\frac{\partial \omega}{\partial r}, \tag{2}$$

where

$$r = \frac{\rho}{at}, \qquad \beta = \frac{b}{a}, \qquad \Delta = \frac{\partial U_\rho}{\partial r} + \frac{U_\rho}{r} + \frac{1}{r}\frac{\partial U_\varphi}{\partial \varphi}, \qquad \omega = \frac{\partial U_\varphi}{\partial r} + \frac{U_\varphi}{r} - \frac{1}{r}\frac{\partial U_\rho}{\partial \varphi}.$$

Figure 3

Figure 3: Figure 3

The solution of these equations must satisfy, for $|\varphi| = \pi/2$, the boundary conditions

$$(1-2\beta^2)\frac{\partial U_\rho}{\partial r} + \frac{1}{r}\frac{\partial U_\varphi}{\partial \varphi} + \frac{U_\rho}{r} = 0,$$

$$\frac{1}{r}\frac{\partial U_\rho}{\partial \varphi} + \frac{\partial U_\varphi}{\partial r} - \frac{U_\varphi}{r} = 0$$

and the condition of continuity of the vector U on the lines of discontinuity.

Fig. 3. Shear strains ω along the lateral surface of the rods. The notation is the same as in Fig. 2. In the one-dimensional approximation ω is a δ-function

The boundary conditions can be rewritten in the form:

$$2\beta^2\frac{\partial U_\rho}{\partial r} - \Delta = 0,$$

$$2\frac{\partial U_\varphi}{\partial r} - \omega = 0.$$

Differentiating these conditions with respect to r and eliminating the functions $\partial^2 U_\rho/\partial r^2$ and $\partial^2 U_\varphi/\partial r^2$ by means of equations (2), we obtain the boundary conditions for the functions Δ and ω:

$$(2\beta^2 - r^2)\frac{\partial \Delta}{\partial r} - \frac{2\beta^4}{r}\frac{\partial \omega}{\partial \varphi} = 0,$$

$$\frac{2}{r}\frac{\partial \Delta}{\partial r} + (2\beta^2 - r^2)\frac{\partial \omega}{\partial r} = 0.$$

In complex form the solution of the equations

$$(1-r^2)\frac{\partial^2 \Delta}{\partial r^2} + \frac{1}{r}(1-2r^2)\frac{\partial \Delta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Delta}{\partial \varphi^2} = 0,$$

$$(\beta^2 - r^2)\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}(\beta^2 - 2r^2)\frac{\partial \omega}{\partial r} + \frac{\beta^2}{r^2}\frac{\partial^2 \omega}{\partial \varphi^2} = 0$$

under the indicated boundary conditions has the form

Fig. 4

Figure 4: Fig. 4

$$\frac{\partial \Delta}{\partial z_1} = \frac{2iv(1-2\beta^2)(2\beta^2 z_1^2 - 1)}{\pi a \left[(2\beta^2 z_1^2 - 1)^2 - 4\beta^3 z_1^2 \sqrt{z_1^2 - 1} \sqrt{\beta^2 z_1^2 - 1} \right] (z_1^2 - 1)},$$

$$\frac{\partial \omega}{\partial z_2} = \frac{-4v z_2 (1-2\beta^2)}{\pi a \left[(2\beta^2 z_2^2 - 1)^2 - 4\beta^3 z_2^2 \sqrt{z_2^2 - 1} \sqrt{\beta^2 z_2^2 - 1} \right] \sqrt{z_2^2 - 1}},$$

where

$$z_1 = \frac{\sin\varphi}{r} + i\cos\varphi \sqrt{\frac{1}{r^2} - 1}, \qquad z_2 = \frac{\sin\varphi}{r} + i\cos\varphi \sqrt{\frac{1}{r^2} - \frac{1}{\beta^2}},$$

with

$$\Delta(r,\varphi) = \operatorname{Re}\Delta(z_1) \quad \text{and} \quad \omega(r,\varphi) = \operatorname{Re}\omega(z_2).$$

The obtained solutions have a singularity at the points $z^2 = 1$ and at the Rayleigh point. The function Δ is bounded at these points, but discontinuous. The function ω is continuous at the point $z = 1$ and unbounded at the Rayleigh point.

Since the equations of elasticity were derived under the condition that the strains Δ and ω are small, the solution obtained is valid only outside a neighborhood of the Rayleigh point.

In the general case, the computation of the functions Δ, ω, u_ρ, and u_φ is cumbersome. Therefore we shall confine ourselves to computing these functions on the lateral surface of the rods. This will make it possible to obtain the change in the shape of the rods during impact and to determine the stress waves along the lateral surface, which is important from the standpoint of experimental verification.

The results of the calculations are presented in Figs. 2–4. The solution found is valid up to the time $t = d/2a$, where d is the thickness of the rods. However, by virtue of the linearity of the equations, the solution for $t < d/a$ can be found by the method of superposition.

Fig. 4: Change in the shape of the rod during collision ($t = d/a,\ \nu = 0.1a$). The notation is the same as in Fig. 2

For $t > d/a$, unloading waves are reflected from the free surfaces. The solution of this problem is also found by the method of S. L. Sobolev; however, it is not considered in the present paper.

Thus, a solution has been obtained for the two-dimensional problem of the elastic collision of rods. According to this solution, at an arbitrarily small collision velocity there exists a region in which the shear strains are arbitrarily large. Since at large shear strains plastic flow occurs, residual deformations caused by this flow always arise when rods collide. On this basis one may conclude that a perfectly elastic impact is impossible. This conclusion is consistent with the known fact that increasing the accuracy of an experiment for determining the limiting velocity of elastic impact leads to a significant decrease in the magnitude of this velocity. In an ideal experiment it would be equal to zero.

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CITED LITERATURE

1. Ph. Frank, R. Mises, *Differential and Integral Equations of Mathematical Physics*, Ch. 12, Moscow–Leningrad, 1937.

Note: Figure translations are in progress. See original paper for figures.