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Abstract

Full Text

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DUAL EXTREMAL PROBLEMS

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In various branches of mathematics (linear programming, game theory, the abstract L-problem of the theory of moments, extremal and approximation questions in the theory of functions), the establishment of criteria for the solution of the extremal problems under study, the investigation of questions of existence and uniqueness, and also the development of algorithms are based on the consideration of extremal problems that are, in a certain sense, dual to those under investigation. This paper sets forth a certain unified approach to the identification of dual problems and gives examples.

1. Let, in a fixed set E, subsets A_τ, B_τ be singled out ($-\infty \leq \alpha < \tau < \beta \leq +\infty$), ordered increasingly ($A_{\tau'} \subset A_{\tau''}, B_{\tau'} \subset B_{\tau''}$ if $\tau' \leq \tau''$), with $A_{\alpha+0} \cap B_{\alpha+0} = \Lambda$, $A = A_{\beta-0} \neq \Lambda$, $B = B_{\beta-0} \neq \Lambda$, where $A_{\tau-0} = \bigcup_{\alpha<\tau'<\tau} A_{\tau'}$, $A_{\tau+0} = \bigcap_{\tau<\tau'<\beta} A_{\tau'}$, Λ is the empty set.

Problem I. Among the admissible elements $a \in A \cap B$, find an optimal one, for which

$$t(a) = \inf\{\tau : a \in A_\tau \cap B_\tau\}$$

attains a minimum.

Remark. The consideration of two chains A_τ, B_τ, rather than one $C_\tau = A_\tau \cap B_\tau$, is connected with the way in which the sets are specified. If, for example, A_τ and B_τ are convex polyhedra specified by their vertices, then it is desirable to avoid determining the vertices of C_τ.

Consider some set E^* of real-valued functions defined on E, and for $\tau \in (\alpha, \beta)$ put:

$$A^*_\tau = \{f \in E^* : f(a) \leq 0 \text{ for } a \in A_{\tau+0}\}, \quad B^*_\tau = \{\varphi \in E^* : \varphi(b) > 0 \text{ for } b \in B_{\tau-0}\}.$$

These sets are, obviously, ordered decreasingly. Denote their unions with respect to τ respectively by A^* and B^*.

Problem I′. Among the admissible elements $f \in A^* \cap B^*$, find an optimal one, for which

$$t^*(f) = \sup\{\tau : f \in A^*_\tau \cap B^*_\tau\}$$

attains a maximum.

It is not difficult to verify that $t(a) \geq t^*(f)$ for any admissible a and f. Therefore, for the problems under consideration the following relations hold:

$$\beta \geq t_0 = \inf\{t(a) : a \in A \cap B\} \geq \sup\{t^*(f) : f \in A^* \cap B^*\} = t_0^* \geq \alpha \quad (1)$$

(where, for $A \cap B = \Lambda$ or $A^* \cap B^* = \Lambda$, one takes respectively $t_0 = \beta$, $t_0^* = \alpha$).

Let E^* be a sufficiently broad set, namely, if in one of the problems there is an admissible element and $A_\tau \cap B_\tau = \Lambda$, then $A^*_{\tau'} \cap B^*_{\tau'} \neq \Lambda$ for every $\tau' < \tau$ (condition (A)). Then equality is attained in the middle inequality (1), except for the obvious case when equality is attained in both extreme inequalities (there is no admissible element in either problem). In this case we shall call Problem I′ **dual** to Problem I.

If (A) is satisfied, for the existence of a solution of problem I or I′ it is necessary that admissible elements exist in both problems. The latter is sufficient for the existence of a solution of problem I if

$$A_{\tau'} \cap B_\tau \neq \Lambda \text{ for all } \tau' > \tau > \alpha \quad \text{implies} \quad \bigcap_{\tau' > \tau} (A_\tau \cap B_{\tau'}) \neq \Lambda, \quad (2)$$

and of problem I′ if

$$A^*_{\tau'} \cap B^*_\tau \neq \Lambda \text{ for all } \tau' < \tau < \beta \quad \text{implies} \quad \bigcap_{\tau' > \tau} (A^*_{\tau'} \cap B^*_\tau) \neq \Lambda. \quad (3)$$

Thus, if (A), (2), and (3) are satisfied, the following is true.

Duality theorem. *If in one of the problems I or I′ there is no admissible element, then neither of them has an optimal one. If, however, admissible elements exist in both, then they also contain optimal elements a_0, f_0, with*

$$\beta > t_0 = t(a_0) = t^*(f_0) = t_0^* > \alpha.$$

The scheme described is a skeleton of the concept being developed, whose main content consists of propositions that make it possible, in concrete problems, to verify the corresponding conditions. In the important case for applications in which A_τ and B_τ are convex sets of a vector space E, one usually takes as the elements of E^* functions of the form $f(a) = L(a) + c$, where L is a linear functional and c is a number; and the main instrument of the analysis is provided by various theorems on the separation of convex sets by hyperplanes*.

2. Let us consider the general problem of linear programming.

Problem II. Among the admissible vectors $x = (x_1, \dots, x_m)$, $x_i \geq 0$, $i = 1, \dots, m$, satisfying the inequalities $\sum_i a_{ij} x_i \geq b_j$, $j = 1, \dots, n$, find an **optimal** one for which $t(x) = \sum_i c_i x_i$ attains its minimum.

Putting $a^i = (a_{i1}, \dots, a_{in}, c_i)$, $b^0 = (b_1, \dots, b_n, 0)$, e^j—the unit vector of axis j, and

$$a(x) = \sum_i x_i a^i - \sum_j x'_j e^j,$$

where

$$x'_j = \sum_i a_{ij} x_i - b_j,$$

we arrive at problem I, in which $\alpha = -\infty$, $\beta = +\infty$, all A_τ coincide with the convex cone spanned by the points $a^1, \dots, a^m, -e^1, \dots, -e^n$, while

$$B_\tau = \{b = b^0 + \tau' e^{n+1} : \tau' \leq \tau\}.$$

Taking for E^* functions defined on the points $a = (a_1, \dots, a_{n+1})$ of the form

$$f(a) = \sum_{j=1}^{n} y_j a_j - a_{n+1},$$

we obtain the dual problem.

Problem II′. Among the admissible vectors $y = (y_1, \dots, y_n)$, $y_j \geq 0$, $j = 1, \dots, n$, satisfying the inequalities $\sum_j a_{ij} y_j \leq c_i$, $i = 1, \dots, m$, find an **optimal** one for which $t^*(y) = \sum_j b_j y_j$ attains its maximum.

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* In the analysis of extremal problems in infinite-dimensional vector spaces E, some generalizations of the known separation theorems connected, for example, with the concept of quasisolidity of a convex set are useful. By the latter is meant the case where the convex set $A \subset E$ contains no more than a countable subset A_0 such that, for any $l \in E$, for some $a_0 \in E_0$ and $\tau > 0$, the point $a_0 + \tau l \in A$.

Theorem. *If the point b is external with respect to the quasisolid convex set A (for any $l \in E$, for some $t > 0$ the segment $\{b + \tau l : \tau \in [0, t]\}$ does not intersect A), then there exists in E a linear functional L such that*

$$\sup\{L(a) : a \in A\} < L(b).$$

Hence follows the possibility of separating by a hyperplane the convex cones K_1 and K_2, if one of them is quasisolid and their difference is not everywhere dense in E, i.e., there exists a point b external with respect to the set of elements $a = a_1 - a_2$, where $a_1 \in K_1$, $a_2 \in K_2$.

The conditions (A), (2), (3) are satisfied here. On the basis of the duality theorem we conclude that, for the optimality of an admissible x, it is necessary and sufficient that there exist an admissible y such that in

$$\sum_i c_i x_i \geqslant \sum_i \left(\sum_j a_{ij} y_j \right) x_i = \sum_j \left(\sum_i a_{ij} x_i \right) y_j \geqslant \sum_j b_j y_j$$

the extreme terms are equal, i.e. $\sum_j a_{ij} y_j = c_i$ for $x_i > 0$, while when $\sum_i a_{ij} x_i > b_j$ the component $y_j = 0$.

The generalized linear-programming problem considered in (1) reduces to Problem I, where A_τ is a fixed cone; however, B_τ is no longer a ray, but convex polyhedra of a more complicated nature. Problems of convex programming ([3], [4]), which essentially consist in finding an extreme point of intersection of a certain axis with a convex (not polyhedral) set of a finite-dimensional space, also fit into the scheme given above. Continuous analogues of problems of linear programming and the theory of matrix games ([3], [5], [6]) lead to Problems I and I′ in infinite-dimensional vector spaces.

3. Let us dwell on the approximation problem, which has been studied from the point of view under consideration by many authors (for the case of approximation by elements of a subspace, first in ([7], [8]), and then, for example, in ([9]); for the case of an arbitrary convex set, in ([10])).

Problem III. In a convex set A of a linear normed space E, find an element a for which the minimum of $t(a) = \|a - b_0\|$ is attained, where b_0 is a fixed element of $E \setminus A$.

In the corresponding Problem I, $A_\tau = A$, $B_\tau = \{b \in E : \|b - b_0\| \leqslant \tau\}$, $\tau \in (0, +\infty)$. Putting $E^* = \{f\}$, where $f(a) = L(a) + c$, L is a linear (continuous) functional, c a number, we arrive at the dual problem.

Problem III′. In E^* find a function f satisfying the inequalities

$$f(a) \leqslant 0 \quad (a \in A), \qquad f(b_0) > 0,$$

for which

$$t^*(f) = \sup\{\tau : f \in B^*_\tau\} = \frac{f(1)}{\|L\|}$$

attains a maximum.

Since conditions (A) and (3) are satisfied here (owing to the existence of a hyperplane separating the set A and the open sphere $B_{\tau-0}$ not intersecting it), the solution a of Problem III is characterized by the existence of an admissible function f for which $t^*(f) = \|a - b_0\|$. This means that the element $a_0 \in A$ is then and only then the least deviating from b_0 when there exists a nontrivial linear functional L satisfying the conditions:

$$1^\circ.\ L(b_0 - a_0) = \|L\|\,\|b_0 - a_0\|, \qquad 2^\circ\ L(a) \leqslant L(a_0) \quad \text{for } a \in A.$$

Remark. Let A be a polyhedron consisting of elements

$$a = \sum_{i=1}^{m} x_i a_i + \sum_{j=1}^{n} y_j b_j + \sum_{k=1}^{p} z_k c_k,$$

where a_i, b_j, c_k are fixed elements of E, x_i are arbitrary coefficients, y_j, z_k are nonnegative, and moreover

$$\sum_k z_k = 1.$$

Condition 2° reduces to the following:

$$2^{00}.\ L(a_i) = 0, \quad i = 1, \dots, m, \qquad L(b_j) \leqslant 0, \quad j = 1, \dots, n,$$

and $L(c_q) = \max L(c_k)$, if the corresponding y_j^0 or z_q^0 are positive.

If, in addition, the functional L is determined uniquely by condition 1° (up to a factor), then testing an element for optimality reduces to checking a finite number of relations (condition 2^{00}). If some of them fail, a way of improving the element under consideration is indicated. On this basis an algorithm for solving the problem is constructed.

Suppose now that on A a convex function $v(a)$ is additionally given and one seeks an element $a \in A$ from the condition of minimizing

$$t(a) = \max\{\|a - b_0\|,\, v(a)\}.$$

In this case B_τ do not change, while $A_\tau = \{a \in A : v(a) \leqslant \tau\}$.

A similar problem, where $t(a) = \|a - b_0\| + v(a)$ (studied in ([2]) for the case when A is a finite-dimensional subspace), leads to Problem I with $A_\tau = \{a = a_1 + a_2 : a_1 \in A,\ a_2 \in E,\ v(a_1) + \|a_2\| \leq \tau\}$, $B_\tau = \{b_0\}$.

Questions of uniform approximation by generalized polynomials, as is known (see ([11])), and also by generalized rational functions ([12]), lead to Problems I and I′ in finite-dimensional spaces, which makes it possible to propose a number of effective methods for their solution.

To the examples given one may add the numerous applications of special types of Problems I and I′, considered by S. Ya. Khavinson, to the analysis of extremal and approximation problems in function theory.

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Note: Figure translations are in progress. See original paper for figures.