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Abstract

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BOUNDARY PROPERTIES OF A CLASS OF MAPPINGS IN SPACE

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Consider a linearly connected metric space R with metric ρ. Following Mazurkiewicz, introduce in R a new metric μ_R, in which the distance $\mu_R(p, q)$ between two points $p \in R$, $q \in R$ is measured by the greatest lower bound of the diameters of arcs $\gamma \subset R$ containing both points p, q.

Definition. A homeomorphic mapping T of one linearly connected space R_1 onto another space R_2 **belongs to the class** $T_\mu(R_1, R_2)$ if, for any two connected sets $L \subset R_1$, $M \subset R_1$ and their images $L^* = T(L)$, $M^* = T(M)$, the equalities $\mu_{R_1}(L, M) = 0$ and $\mu_{R_2}(L^*, M^*) = 0$ are equivalent.

If R_1 is a ball R in N-dimensional ($N \geqslant 2$) Euclidean space E^N, and R_2 is a certain domain $D \subset E^N$ homeomorphic to it, considered in the spherical metric*, then the union of the mappings belonging to the classes $T_\mu(R, D)$ will be denoted by $T_{\tilde{\mu}}$.

We study the boundary properties of mappings of the class $T_{\tilde{\mu}}$.

In what follows the following notation will be adopted: $\tilde{\mu}_D$ is the metric μ_D corresponding to the spherical metric in the domain D; $d(M)$ is the diameter of the set M in the spherical metric; $\operatorname{lt} M_n$ is the upper topological limit of the sequence of sets M_n ($n = 1, 2, ...$); ∂D is the boundary of the domain D; $M^* = T(M)$ is the image of the set M under the mapping T. By an arc we shall mean the homeomorphic image of a segment, and by a path γ the homeomorphic image $\gamma : p = p(t)$ of the half-interval $[0 \leqslant t < 1)$. We say that the path γ goes to the point p_1 if $p(t) \to p_1$ as $t \to 1$.

In the domain $D \subset E^N$ consider the set of paths each of which goes to some point of the boundary of the domain D. Paths $\gamma_1 \subset D$ and $\gamma_2 \subset D$ are considered D-equivalent if and only if they go to one and the same point of the boundary D and in any neighborhood of this point they can be joined by an arc $\lambda \subset D$ not going outside this neighborhood.

Definition. A class of D-equivalent paths is called an **attainable point** of the boundary of the domain D.

It is convenient to write an attainable point in the form of a pair (p, γ), where γ is an arbitrary representative of the class of D-equivalent paths, and p is the point of the boundary D to which the paths of this class go.

Theorem 1. *Under a mapping $T \in T_{\tilde{\mu}}$ of the ball R onto the domain D^*, to each attainable boundary point (p^*, γ^*) of the domain D^* one can assign a point p of the boundary of R such that:*

a) the preimage of any path γ^* determining the attainable point (p^*, γ^*) goes to the point p;

b) distinct attainable boundary points of the domain D^* correspond to distinct points of the boundary sphere S;

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* This has to be done in order not to exclude from consideration mappings onto unbounded domains.

c) every continuum $\Gamma \subset S$ not degenerating to a point contains a point corresponding to some attainable boundary point of the domain D^*.

From the results of paper ([6]) it follows that quasiconformal and, in particular, conformal mappings of a ball $R \subset E^N$ $(N \geq 2)$ belong to the class $T_{\tilde{\mu}}$; therefore, in the planar case, if T is a conformal mapping of a disk, Theorem 1 becomes the well-known theorem of Kœbe ([4]) on the correspondence of attainable points under conformal mappings of a disk.

For the proof of Theorem 1 we shall need the following four lemmas.

Lemma 1. For every mapping T of a ball R of the class $T_{\tilde{\mu}}$ there exists a nondecreasing function $\varphi(d)$, $d > 0$, possessing the following properties:

a) $\varphi(d) > 0$ for $d > 0$;

b) $\varphi(d) \to 0$ as $d \to 0$;

c) for any arc $\gamma \subset R$, from the fact that $d(\gamma^*) < \varphi(d)$ it follows that $d(\gamma) < d$.

Lemma 2 (on an ε-net). If T is a mapping of the class $T_{\tilde{\mu}}$ of a ball R onto some domain D, and $\{\gamma_\alpha\}$ is an arbitrary family of arcs in R with diameters uniformly separated from zero, $d(\gamma_\alpha) \geq d_0 > 0$, then in the family $\{\gamma^*_\alpha\}$ of their images, for any $\varepsilon > 0$, one can select a finite ε-net, i.e. such a finite system $\Omega = \{\gamma^*_{\alpha_1}, \dots, \gamma^*_{\alpha_{j(\varepsilon)}}\}$ of arcs that for any arc $\gamma^*_\alpha \in \{\gamma^*_\alpha\}$ there is an arc $\gamma^*_{\alpha_\nu} \in \Omega$ for which $\tilde{\mu}_D(\gamma^*_\alpha, \gamma^*_{\alpha_\nu}) < \varepsilon$.

Definition. Let T be a mapping of a domain D. The **oscillation of the mapping** T along a path $\gamma : p = p(t)$, $\gamma \subset D$, is the quantity

$$\omega(T, \gamma) = \lim_{\tau \to 1} d(\gamma^*_\tau),$$

where γ^*_τ is the image of the path $\gamma_\tau : p = p(t)$ $[0 \leq \tau \leq t < 1]$ under the mapping T.

Lemma 3. Let $\{\gamma_n\}$ be a sequence of arcs in a ball R with diameters uniformly separated from zero: $d(\gamma_n) \geq d_0 > 0$ $(n = 1, 2, ...)$. If, under the mapping $T \in T_{\tilde{\mu}}$ of the ball, $d(\gamma_n^*) \to c$ as $n \to \infty$, then for any point $p \in \overline{\bigcup \gamma_n}$ one can construct a path $\gamma \subset R$, tending to the point p, such that $\omega(T, \gamma) \leq 2c$.

Lemma 4. If Γ is a continuum on the boundary of the ball R distinct from a point; T is a mapping of the class $T_{\tilde{\mu}}$, and γ is a path in R tending to some point $\alpha \in \Gamma$, then for any $\varepsilon > 0$ there exists a path $\tilde{\gamma} \subset R$, tending from some point of the path γ to some point $\tilde{\alpha} \in \Gamma$, such that $d(\tilde{\gamma}^*) < \varepsilon$.

It is interesting to note that the assertions of each of Lemmas 2, 3, 4 and of the theorem are equivalent for mappings $T \in T_{\tilde{\mu}}$.

Definition. A sequence $\{D_n\}$ of subdomains of a domain D is called **regular** if: a) $D_{n+1} \subset D_n$; b) $\left[\bigcap_{n=1}^{\infty} \overline{D_n}\right] \subset \partial D$; c) the relative boundary g_n in D of any domain $D_n \in \{D_n\}$ is connected; d) $\tilde{\mu}_D(g_n, g_{n+1}) > 0$ $(n = 1, 2, ...)$; e) there is no more than one attainable boundary point of the domain D which is an attainable boundary point of each of the domains of the sequence.

Theorem 2. In order that, under a homeomorphic mapping of a ball $R \subset E^N$ $(N \geq 2)$ onto a domain D, every regular sequence of subdomains of R pass into a regular sequence of subdomains of D, and the preimage of every regular sequence of subdomains of D be a regular sequence of subdomains of R, it is necessary and sufficient that $T \in T_{\tilde{\mu}}$.

Definition. We shall say that a sequence of domains $\{D'_n\}$ is **embedded** in the sequence of domains $\{D''_n\}$ if every member of the sequence $\{D''_n\}$ contains all members of the sequence $\{D'_n\}$.

with sufficiently large indices. Two sequences of domains nested in one another will be called **equivalent**.

Lemma 5. A regular sequence of subdomains of the ball contracts to a point (i.e. the intersection

$$\Gamma = \bigcap_{n=1}^{\infty} \overline{D}_n$$

consists of one point), and two regular sequences of subdomains of the ball are equivalent if and only if they contract to one and the same point of the boundary sphere.

Definition. A class of equivalent regular sequences of subdomains of a domain D is called a **boundary element** of the domain D.

It is convenient to write a boundary element in the form of a pair $(\Gamma, \{D_n\})$, where $\{D_n\}$ is an arbitrary element of the class of equivalent regular sequences, and

$$\Gamma = \bigcap_{n=1}^{\infty} \overline{D}_n$$

is the continuum on ∂D determined by this class.

From Theorem 2, taking Lemma 5 into account, it follows that

Theorem 3. Under a mapping $T \in T_{\tilde{\mu}}$ of the ball R onto the domain D^*, between the points of the boundary sphere S and the boundary elements of the domain D^* one can establish a one-to-one correspondence, under which the boundary element $(\Gamma^*, \{D_n^*\})$ of the domain D^* corresponds to the point on S determined by the regular sequence $\{D_n\} = T^{-1}(\{D_n^*\})$.

From this theorem one obtains the following two theorems, which give, respectively, criteria for continuous and homeomorphic extendability of a mapping $T \in T_{\tilde{\mu}}$ to $\overline{R}$.

Theorem 4. In order that a mapping $T \in T_{\tilde{\mu}}$ of the ball R onto the domain D^* extend continuously to $\overline{R}$, it is necessary and sufficient that every regular sequence of subdomains of D^* determine on ∂D^* only one point.

Theorem 5. In order that a mapping $T \in T_{\tilde{\mu}}$ of the ball R onto the domain D^* extend homeomorphically to $\overline{R}$, it is necessary and sufficient that every regular sequence of subdomains of D^* determine on ∂D^* only one point, and that the points determined by nonequivalent regular sequences be distinct.

Using the last theorem, it is easy to construct examples of spatial domains homeomorphic to the ball which are not equivalent to the ball with respect to mappings of the class $T_{\tilde{\mu}}$. This circumstance substantially distinguishes the spatial case from the planar one. For quasiconformal mappings, the first examples of such domains were constructed by B. V. Shabat ([5]), Väisälä ([1]), and Gehring ([2]). The domains appearing in the cited works cannot be mapped onto the ball not only quasiconformally, but also by a mapping of the class $T_{\tilde{\mu}}$. We note, however, that by using the method of moduli one can construct a spatial Jordan domain which cannot be mapped quasiconformally onto the ball, whereas with the aid of mappings of the class $T_{\tilde{\mu}}$ a Jordan domain can, of course, always be mapped onto the ball.

We give one necessary condition for a domain D to be transformable into the ball by a mapping of the class $T_{\tilde{\mu}}$.

Definition. A regular sequence $\{D_n\}$ of subdomains of a domain D is called **minimal** if every regular sequence of subdomains of D nested in it is equivalent to $\{D_n\}$.

From Theorem 2 and Lemma 5 one obtains

Theorem 6. In order that a domain D be equivalent to the ball with respect to mappings of the class $T_{\tilde{\mu}}$, it is necessary that every regular sequence of subdomains of D be minimal.

As simple examples show, this necessary condition is not sufficient.

Let us note some features of mappings of the class $T_{\widetilde{\mu}}$ in the plane. In the planar case, the boundary elements introduced above are topologically equivalent to simple ends in the sense of Carathéodory; therefore Theorem 3 may be regarded as a generalization of Carathéodory' s theorem [3] on boundary correspondence under conformal mappings. Moreover, it follows from Theorem 2 that, in the plane, $T_{\widetilde{\mu}}$ is the broadest class of homeomorphic mappings of the disk for which boundary correspondence in the sense of Carathéodory' s simple ends is realized.

In the planar case the following refinement of Theorem 1 also holds.

Theorem 7. *For a homeomorphic mapping T of the disk, the assertions of the Koebe theorem hold if and only if $T \in T_{\widetilde{\mu}}$.*

Hence, taking Theorem 2 into account, we obtain

Theorem 8. *For a homeomorphic mapping of the disk, the assertions of the Koebe theorem hold if and only if, for it, the assertions of the Carathéodory theorem are fulfilled.*

Thus, the assertions of the Koebe theorem and the Carathéodory theorem are equivalent.

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Note: Figure translations are in progress. See original paper for figures.