



Soviet-era science, translated into English

Reports of the Academy of Sciences of the USSR

O. N. GOLOVIN

1963

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196301.22917>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1963. Volume 153, No. 6

MATHEMATICS

O. N. GOLOVIN

THE STRUCTURE OF POLYVERBAL OPERATIONS

(Presented by Academician A. I. Mal'cev on 3 VII 1963)

In work ⁽¹⁾ I introduced the concepts of a polyverbal subgroup and a polyverbal operation and formulated a number of theorems concerning, for the most part, not all such subgroups and operations, but only neutral ones. A detailed exposition of these results and their further development (especially in the direction of studying the associativity of operations) was given by me in ⁽²⁾. In the present paper the set of all polyverbal subgroups and operations is studied, and not only the neutral ones. The terminology and notation used below coincide with those used in ⁽²⁾.

1. It is known that between the varieties of groups and the verbal subgroups of a free group of countable rank there is a natural one-to-one correspondence. This correspondence is an inverse isomorphism between the complete (completely Dedekind) structures of varieties and verbal subgroups (both ordered by set-theoretic inclusion). In an analogous way there exists (see ⁽²⁾, pp. 416-419) a natural one-to-one correspondence τ between polyverbal subgroups W of the base group

$$X = \prod_{i=1}^{\infty} * X_i,$$

where each X_i is a free group of countable rank, and polyverbal operations Π^W on the class of all groups.

For a free group, the concepts of a verbal and a fully characteristic subgroup are equivalent. For the base group X , its subgroup W is polyverbal if and only if it is invariant and withstands all endomorphisms assembled from endomorphisms of the factors X_i , and all substitutions on the set of these factors. Therefore the set \mathfrak{L} of all polyverbal subgroups of the base group is also a complete (completely Dedekind) structure with X as unit and E as zero. The same assertion is true also for the set \mathfrak{P} of all polyverbal operations, if one regards $\Pi^{W_1} \leq \Pi^{W_2}$ if

and only if Π^{W_1} is a factor-operation of Π^{W_2} (which, in turn, is equivalent to the inclusion $W_1 \supseteq W_2$). The unit in \mathfrak{P} is the free product $\Pi^* = \Pi^E$, and the zero is the zero product $O = \Pi^X$, which assigns to any set of groups the trivial group E . The mapping $\tau : W \rightarrow \Pi^W$ is an inverse isomorphism between the structures \mathfrak{L} and \mathfrak{P} .

Each polyverbal subgroup W of the base group X is regularly embedded in it. This means (see ⁽³⁾, pp. 439-440 and ⁽⁴⁾, p. 587) that it is representable in the form of a product:

$$W = \prod_{i=1}^{\infty} W_i \cdot W_N, \quad (1)$$

where $W_i = W \cap X_i$, and $W_N = W \cap C$, where $C = [X_i]^X$ is the Cartesian subgroup of the product, i.e. the normalized mutual commutant of the factors X_i in X .

The proper components W_i are all isomorphic to one another and are verbal subgroups (over one and the same set of words) of the corresponding X_i . Thus, notation (1) can be refined:

$$W = \prod_{i=1}^{\infty} W_1(X_i)_V \cdot W_N, \quad (2)$$

where $W_1(X_i)_V$ denotes the verbal subgroup of the group X_i determined by the set of words W_1 . We shall call the subgroup W_1 the *verbal*, and W_N the *neutral*, subgroup of W .

By the uniqueness of the proper notation (2), two polyverbal subgroups coincide if and only if their verbals and neutrals respectively coincide. Among all polyverbal subgroups with a given verbal W_1 , there exists a unique minimal one, contained in all the others with the same verbal. This is

$$W_m = W_1(X)_{VV} = \{W_1(X_i)_V \mid i = 1, 2, \dots\}^X = \prod_{i=1}^{\infty} W_1(X_i)_V \cdot W_{mN}.$$

(The notation $W_1(X)_{VV}$ means that the W_1 -polyverbal subgroup is taken from X .)

Theorem 1. *There exists a natural one-to-one correspondence between all polyverbal subgroups W of the base group X and all possible pairs $\langle W_1, W_N \rangle$, where W_1 is an arbitrary verbal subgroup (of the factor X_1), and W_N is a neutral polyverbal subgroup satisfying the condition*

$$W_N \supseteq W_1(X)_{VVN} = W_1(X)_{VV} \cap C. \quad (3)$$

Theorem 2. *Under multiplication (intersection) of any set of polyverbal subgroups of the base group, their verbals and neutrals are respectively multiplied (intersected).*

Of special interest are certain special subsets of the set Ω , for which I introduce the following notation: Ω_N is the set of all neutral polyverbal subgroups, i.e. polyverbal subgroups lying in C ; Ω_V is the set of all verbal subgroups of the group X ; Ω_{VN} is the set of all neutrals of verbal subgroups; Ω_A is the set of all polyverbal subgroups defining associative operations; Ω_{AN} is the set of all neutral polyverbal subgroups defining associative operations; $\Omega_{V(C)}$ is the set of all verbal subgroups of the Cartesian subgroup C ; $\Omega_{V(\tilde{C})}$ is the set of all verbal subgroups of some fixed polyverbal subgroup \tilde{C} of the group X ; Ω_{W_1} is the set of all polyverbal subgroups with given verbal W_1 ; $\Omega_{W_1}^V$ is the set of all polyverbal subgroups with given verbal W_1 whose neutrals are the neutrals V_N of verbal subgroups satisfying the condition $V_N \supseteq W_1(X)_V \cap C$; Ω^V is the set of all polyverbal subgroups of the preceding type for all possible verbals W_1 , i.e.

$$\bigcup_{W_1} \Omega_{W_1}^V.$$

To each of the above-listed subsets of the structure Ω there corresponds, by virtue of τ , a subset of the structure \mathfrak{P} ; we shall denote it analogously, replacing the letter Ω by \mathfrak{P} .

Theorem 3. *The structure \mathfrak{P} of all polyverbal operations decomposes into pairwise disjoint complete substructures ${}^{\tau}\mathfrak{P}_{W_1}$, each of which is isomorphic to some complete substructure of the structure \mathfrak{P}_N of all neutral polyverbal operations. Complete substructures in \mathfrak{P} are—*

* The assertion on the product of neutrals for the case of verbal subgroups was originally proved by O. N. Makedonskaya.

and subsemigroups \mathfrak{P}_V , \mathfrak{P}_{VN} , $\mathfrak{P}_{V(C)}$, $\mathfrak{P}_{V(\tilde{C})}$, $\mathfrak{P}_{W_1}^V$, \mathfrak{P}^V ; the subsemigroups \mathfrak{P}_A and \mathfrak{P}_{AN} of all associative and neutral associative operations are (in any case) its lower complete subsemilattices.

Moran ^(4, 5) proved the inclusions: $\mathfrak{P}_V \subseteq \mathfrak{P}_A$ and $\mathfrak{P}_{VN} \subseteq \mathfrak{P}_{AN}$. The question whether the second inclusion is proper remains open to this day: all neutral associative operations constructed up to now which differ from Moran's verbal ones are not polyverbal. As for the first inclusion, it is certainly improper, since the following assertion holds, establishing new series of associative (non-neutral) polyverbal operations:

Theorem 4. *All operations from \mathfrak{P}^V are associative, i.e. $\mathfrak{P}^V \subseteq \mathfrak{P}_A$.*

Since \mathfrak{P}_A is a complete lower subsemilattice (this means that the intersection of any set of associative operations is associative), every polyverbal operation Π^W

possesses a uniquely determined **associative closure**—the minimal associative operation whose factor-operation is Π^W . The complete substructure \mathfrak{P}^V lies in \mathfrak{P}_A . One may therefore also speak of the \mathfrak{P}^V -closure (and of the \mathfrak{P}^V - “skeleton”) of any operation Π^W , and these are again associative; it is clear that the general associative closure does not exceed the \mathfrak{P}^V -closure. As a rule, both closures of an operation from \mathfrak{P}_{W_1} themselves belong to the same \mathfrak{P}_{W_1} . However, say, the operation Π^{W_m} (where W_m is the minimal subgroup with the given verbal W_1), which is nonassociative when $W_1 \neq E$ and $W_1 \neq X_1$ (i.e. when $\Pi^{W_m} \neq \Pi^*$, $\Pi^{W_m} \neq \Pi^X = O$), cannot have closures in \mathfrak{P}_{W_1} , in view of the fact that it itself is the unit of the structure \mathfrak{P}_{W_1} . What its associative closure is I do not know; however, its \mathfrak{P}^V -closure is free multiplication.

The following two theorems show that, in a certain sense, the transition from considering neutral polyverbal operations to considering arbitrary polyverbal operations cannot be of significant independent interest and cannot help (as might have been expected), for example, in the search for perfect exact operations (see ⁽⁶⁾).

Theorem 5. *Whatever the polyverbal subgroup (2) may be, applying the operation Π^W determined by it to an arbitrary set of groups G_ν , $\nu \in I$, is equivalent to applying a neutral operation Π^{W_N} to their factor-groups $G_\nu/W_1(G_\nu)_\nu$.*

A special case of this theorem is the main part of the assertion of Moran’ s Theorem 7.2 ⁽⁵⁾ (cf. also ⁽⁷⁾, pp. 304–305).

Theorem 6. *All neutral operations different from the direct product and defined by neutrals of non-neutral polyverbal subgroups fail to satisfy Mal’ cev’ s postulate (see ⁽⁶⁾).*

That other operations exist follows from the fact, proved by A. L. Shmel’ kin (relying on the free basis of the Cartesian subgroup effectively given by Grünberg ⁽⁸⁾), that all operations from $\mathfrak{P}_{V(C)}$ satisfy Mal’ cev’ s postulate (for the case of two factors, see ⁽⁹⁾).

2. G. Neumann in ⁽¹⁰⁾ introduced an associative operation of multiplying varieties of groups, with respect to which the semigroup of varieties is free if one removes from it the variety of all groups and the variety consisting of the single group E (see ^(11, 12)).

One can introduce an analogous operation \circ also in the set \mathfrak{P} , setting, by definition,

$$\Pi^W \circ \Pi^{\widetilde{W}} := \Pi^{W(\widetilde{W})_\nu}.$$

Under this operation \mathfrak{P} turns out to be a partially ordered (with respect to the order considered above) semigroup with Π^* as zero

and $O = \Pi^X$ as a left, but not a right, identity. The subsets \mathfrak{P}_N , \mathfrak{P}_V , $\mathfrak{P}_{V(\bar{C})}$ are left (but not right) ideals in \mathfrak{P} , in contrast to \mathfrak{P}_A , \mathfrak{P}_{AN} , and \mathfrak{P}_{VN} , which are not

even subsemigroups. Thus, for example, the \circ -square of direct multiplication is already a nonassociative ⁽¹³⁾ operation $\Pi^{C'}$, where C' is the commutant of the subgroup C . The semigroup \mathfrak{P} contains a free semigroup of varieties of groups; however, throughout \mathfrak{P} even the right cancellation law fails. Thus, for example, $\Pi^C \circ \Pi^C = \Pi^{X'} \circ \Pi^C$.

3. An exact characterization of the place occupied, among all exact (see ⁽²⁾, p. 418) operations, by the neutral polyverbal operations is given by my Theorem 2 from ⁽²⁾. T. M. Baranovich transferred this and other principal results of ⁽²⁾ from groups to universal algebras. In doing so she established that the rejection of the assumption of neutrality of the polyverbal operations under consideration is equivalent to replacing the requirement of "exactness" of the operations by their "semiexactness." This means that one must assume not an isomorphic but, generally speaking, only a homomorphic embedding of the factors into their product. And then the aforementioned Theorem 2 remains valid under a corresponding modification of the postulates entering into its formulation. Analogues of most of the other theorems from ⁽²⁾ are also valid, in particular Theorems 5, 6, 7, and 8. In the case of groups, the proofs of the theorems are also essentially preserved.

Moscow State University
named after M. V. Lomonosov

Received
1 VII 1963

REFERENCES

- ¹ O. N. Golovin, DAN, **145**, No. 5, 967 (1962).
- ² O. N. Golovin, Tr. Mosk. matem. obshch., **12**, 413 (1963).
- ³ O. N. Golovin, Matem. sborn., **27**(69), No. 3, 427 (1950).
- ⁴ S. Moran, Proc. London Math. Soc. (3), **6**, No. 24, 581 (1956).
- ⁵ S. Moran, Proc. London Math. Soc. (3), **8**, No. 32, 548 (1958).
- ⁶ O. N. Golovin, DAN, **149**, No. 1, 12 (1963).
- ⁷ S. Moran, Proc. London Math. Soc. (3), **9**, No. 34, 287 (1959).
- ⁸ K. W. Gruenberg, Proc. London Math. Soc. (3), **7**, No. 25, 29 (1957).
- ⁹ A. L. Shmelkin, Matem. sborn., **51**(93), No. 3, 277 (1960).
- ¹⁰ H. Neumann, Math. Zs., **65**, No. 1, 36 (1956).
- ¹¹ B. H. Neumann, H. Neumann, P. Neumann, Math. Zs., **80**, No. 1, 44 (1962).
- ¹² A. L. Shmelkin, DAN, **149**, No. 3, 543 (1963).
- ¹³ R. R. Struik, Trans. Am. Math. Soc., **81**, No. 2, 425 (1956).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.