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Abstract

Full Text

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ON THE BEST APPROXIMATION OF ANALYTIC FUNCTIONS IN THE MEAN OF FIRST DEGREE ON A CIRCLE

(Presented by Academician V. I. Smirnov on 6 V 1963)

1°. We shall consider the problem of determining exact and asymptotic values for the best approximation in the metric L on the circle $|z| = 1$ of the analytic functions $\frac{1}{z-a}$, $\ln(z-a)$, $(z-a)^s$, and other functions, by polynomials of degree n . For the best approximation of the indicated functions in the metric L on the segment $[-1, +1]$ of the real axis for real a , such a question was studied in the works of S. M. Nikol'skii⁽¹⁾, N. I. Akhiezer⁽²⁾, and others. The best approximation in the metric L of a function $f(z) \in L$ on the circle $|z| = r$, $0 < r \leq 1$, will be denoted by $\rho_n^{(1)}[f, r]$:

$$\rho_n^{(1)}[f, r] = \inf \int_{|z|=r} |f(z) - Q_n(z)| \cdot |dz|$$

over all possible polynomials $Q_n(z)$ of degree n ; $\rho_n^{(1)}[f, 1] = \rho_n^{(1)}[f]$.

We shall use the following criterion:

In order that $P_n(z)$ be a polynomial of best approximation in the metric L_p , $p \geq 1$, of degree n to the function $f(z) \in L_p$ on a rectifiable curve Γ^* , it is sufficient and (for $p = 1$ and the case where the difference $f(z) - P_n(z)$ is nonzero almost everywhere on Γ) necessary that, for every polynomial $Q_n(z)$ of degree n , the equality

$$\int_{\Gamma} |f(z) - P_n(z)|^{p-1} \operatorname{Re}\{Q_n(z) \operatorname{sign}[\overline{f(z) - P_n(z)}]\} dz = 0$$

hold, where $\operatorname{sign} w = w/|w|$ for $w \neq 0$ and $\operatorname{sign} w = 0$ for $w = 0$.

In the case when Γ is the segment $[-1, +1]$, $f(z)$ is a real-valued function, and $p = 1$, this criterion was established by S. M. Nikol'skii⁽³⁾.

In the general case this criterion is an analogue of the criterion established by A. F. Timan⁽⁴⁾ with the metric L_p defined by an integral over a domain. In the case when $p = 1$ and Γ is the circle $|z| = 1$, the polynomial $P_n(z)$ of degree n gives the best approximation if the equality

$$\int_{|z|=1} z^{-m} \frac{f(z) - P_n(z)}{|f(z) - P_n(z)|} |dz| = 0 \tag{1}$$

holds for all $m = 0, 1, 2, \dots, n$.

2°. The starting point is the determination of the quantity $\rho_n^{(1)} \left[\frac{1}{z-a} \right]$, $|a| \neq 1$.

Consider the identity

$$\frac{1}{z-a} - P_n(z) = M \frac{1 + \bar{a}z + \bar{a}^2 z^2 + \dots + \bar{a}^{n+1} z^{n+1}}{z-a}, \tag{2}$$

* The condition $f(x) \in L_p$ on Γ means that $\int_{\Gamma} |f(z)|^p |dz| < \infty$.

in which $P_n(z)$ is a polynomial of degree n , and the number M is determined from the condition that the residue of the right-hand side at the point $z = a$ is equal to 1, $M = (|a|^2 - 1)(|a|^{2n+4} - 1)^{-1}$. We shall show that $P_n(z)$ is the polynomial of best approximation in the metric L on the circle $|z| = 1$ for the function $\frac{1}{z-a}$. Multiplying the numerator and denominator of the right-hand side of (2) by $\bar{z} - \bar{a}$, where $|z| = 1$, we represent this formula in the form

$$\frac{1}{z-a} - P_n(z) = M \frac{\bar{z} - \bar{a}^{n+2} z^{n+1}}{1 - 2|a| \cos(\vartheta - \alpha) + |a|^2},$$

where $z = e^{i\vartheta}$, $a = |a|e^{i\alpha}$. Hence we find

$$\left| \frac{1}{z-a} - P_n(z) \right| = M \frac{\sqrt{1 - 2|a|^{n+2} \cos(n+2)(\vartheta - \alpha) + |a|^{2n+4}}}{1 - 2|a| \cos(\vartheta - \alpha) + |a|^2}.$$

According to condition (1), it is enough to show that the integral

$$\int_0^{2\pi} \frac{\bar{z}^{n+1} - \bar{a}^{n+2} z^{n+1-m}}{\sqrt{1 - 2|a|^{n+2} \cos(n+2)(\vartheta - \alpha) + |a|^{2n+4}}} d\vartheta \tag{3}$$

vanishes for all $m = 0, 1, 2, \dots, n$.

We note that for $k = 1, 2, \dots, n+1$ the equality

$$\int_0^{2\pi} \frac{\cos k\varphi}{\sqrt{1 - 2c \cos(n+2)\varphi + c^2}} d\varphi = 0, \tag{4}$$

holds, where c is any number > 0 and $\neq 1$. Indeed, this follows from the Fourier expansion

$$(1 - 2c \cos \varphi + c^2)^{-1/2} = \sum_{m=0}^{\infty} A_m \cos m\varphi,$$

in which φ is replaced by the quantity $(n + 2)\varphi$. From (4) there follows the vanishing of the integral (3) for $m = 0, 1, 2, \dots, n$.

Thus, $P_n(z)$ is the polynomial of best approximation, and for $|a| \neq 1$ the formula is valid

$$\rho_n^{(1)} \left[\frac{1}{z-a} \right] = \frac{|a|^2 - 1}{|a|^{2n+4} - 1} = \int_0^{2\pi} \frac{\sqrt{1 - 2|a|^{n+2} \cos(n+2)\vartheta + |a|^{2n+4}}}{1 - 2|a| \cos \vartheta + |a|^2} d\vartheta. \quad (5)$$

From formula (5), for $|a| > 1$, there follows the asymptotic formula

$$\rho_n^{(1)} \left[\frac{1}{z-a} \right] \sim \frac{2\pi}{|a|^{n+2}}. \quad (6)$$

For $|a| < 1$, from (5) it follows that

$$\lim_{n \rightarrow \infty} \rho_n^{(1)} \left[\frac{1}{z-a} \right] = 2\pi. \quad (7)$$

In particular, $\rho_n^{(1)}[z^{-1}] = 2\pi$, and the polynomial of best approximation in mean of the first degree is the identically zero polynomial.

3°. By an analogous method one can find $\rho_n^{(1)} \left[\frac{z^s}{z^p - a^p} \right]$, where p is a natural number > 1 , s is equal to one of the numbers $0, 1, 2, \dots, p-1$, and $|a| \neq 1$. For this purpose the identity

$$\frac{z^s}{z^p - a^p} - Q_n(z) = \frac{|a|^{2p} - 1}{|a|^{2p(k+2)} - 1} H(z), \quad (8)$$

in which $Q_n(z)$ is some polynomial of degree n ; n is any number of the form $kp + s$ (k is an integer ≥ 0) and

$$H(z) = \frac{z^s [1 + \bar{a}^p z^p + \bar{a}^{2p} z^{2p} + \dots + \bar{a}^{(k+1)p} z^{(k+1)p}]}{z^p - a^p}.$$

From identity (8) one obtains the formula

$$\rho_n^{(1)} \left[\frac{z^s}{z^p - a^p} \right] = \frac{|a|^{2p} - 1}{|a|^{2p(k+2)} - 1} \int_0^{2\pi} \frac{\sqrt{1 - 2|a|^{p(k+2)} \cos(k+2)p\vartheta + |a|^{2p(k+2)}}}{1 - 2|a|^p \cos p\vartheta + |a|^{2p}} d\vartheta, \quad (9)$$

which is valid for the numbers $n = kp + s$, and also for $kp + s + 1, \dots, kp + s + p - 1$; $Q_n(z)$ will be a polynomial of best approximation in the mean of degree $n, n + 1, \dots, n + p - 1$. For $|a| > 1$, (9) gives the asymptotic formula

$$\rho_n^{(1)} \left[\frac{z^s}{z^p - a^p} \right] \sim \frac{2\pi}{|a|^{p(k+2)}}$$

as $k \rightarrow \infty$ and $n = kp + s, kp + s + 1, \dots, kp + s + p - 1$. For $|a| < 1$ we obtain

$$\lim_{n \rightarrow \infty} \rho_n^{(1)} \left[\frac{z^s}{z^p - a^p} \right] = 2\pi.$$

In particular, for $a = 0, s = 0$ we shall have $\rho_n^{(1)}[1/z^p] = 2\pi$.

4°. Starting from identity (2), with the aid of integration with respect to the parameter a (first taking a real, $a > 1$), one can obtain the asymptotic formula

$$\rho_n^{(1)}[\ln(z - a)] \sim \frac{2\pi}{n|a|^{n+1}}, \quad (10)$$

valid for any complex a satisfying $|a| > 1$.

5°. If $f(z) = (a - z)^s$ for real s and $a > 1$, then construct a polynomial $P_n(z)$ of degree n which coincides with $f(z)$ at the zeros of the polynomial

$$R(z) = 1 + az + a^2z^2 + \dots + a^{n+1}z^{n+1}.$$

Then

$$(a - z)^s - P_n(z) = \frac{R(z)}{2\pi i} \int_{|\zeta|=r} \frac{(a - \zeta)^s}{(\zeta - z)R(\zeta)} d\zeta,$$

where $1 < r < a$. First assuming $s > -1$ and deforming the contour of integration in the appropriate way, we obtain

$$(a - z)^s - P_n(z) = -\frac{(1 - a^{n+2}z^{n+2}) \sin \pi s}{(1 - az)\pi} \int_a^\infty \frac{(x - a)^s(1 - ax)}{(x - z)(1 - a^{n+2}x^{n+2})} dx.$$

Estimating the integral with the help of S. N. Bernstein' s asymptotic formula (5) and using the result from 2°, we obtain

$$\rho_n^{(1)}[(z-a)^s] \sim \frac{2\Gamma(s+1)|\sin \pi s|}{n^{s+1}|a|^{n-s+1}}. \quad (11)$$

This formula is valid for any complex a , $|a| > 1$, and any real s different from zero and from a positive integer. Setting $s = -p$, it can be represented in the form

$$\rho_n^{(1)}\left[\frac{1}{(z-a)^p}\right] \sim \frac{2\pi n^{p-1}}{\Gamma(p)|a|^{n+p+1}}. \quad (12)$$

If

$$f(z) = \sum_{m=1}^k \frac{A_m}{(z-a)^m} + \varphi(z),$$

where $|a| > 1$, $A_k \neq 0$, and $\varphi(z)$ is analytic in the disk with center $z = 0$ and radius $> |a|$, then

$$\rho_n^{(1)}[f] \sim |A_k| \frac{2\pi n^{k-1}}{(k-1)!|a|^{n+k+1}}.$$

Let us note that the passage to the limit as $a \rightarrow 1$ in formula (11) (for $s > -1$) or in formula (10) cannot be carried out, since the proof of these formulas does not imply their uniformity with respect to a .

6°. The following analogue holds of S. N. Bernstein' s formula (7), proved by him for uniform approximation on the interval $[-1, +1]$:

$$\rho_n^{(1)}[(z-a)^s \Phi(z)] \sim |\Phi(a)| \rho_n^{(1)}[(z-a)^s], \quad (13)$$

where $|a| > 1$, s is not equal to 0 or to a positive integer, and $\Phi(z)$ is an analytic function in some disk $|z| < R$, $R > |a|$, under the condition $\Phi(a) \neq 0$.

7°. Let $H_1^{(s)}$ denote the class of all functions analytic in the disk $|z| < 1$ for which

$$\lim_{\rho \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} |f^{(s)}(\rho e^{i\theta})| d\theta \leq 1,$$

where s is an integer ≥ 0 , $H_1^{(0)} = H_1$. Using the results of K. I. Babenko (7), one can obtain an exact formula for the best mean approximation of functions of the class $H_1^{(s)}$.

Theorem. For $0 < r \leq 1$, when $n \geq s$, the formula

$$\sup_{f \in H_1^{(s)}} \rho_{n-1}^{(2)}[f, r] = 2\pi r^{n+1} \alpha_{n,s},$$

holds, where

$$\alpha_{n,s} = \frac{1}{n(n-1) \dots (n-s+1)}$$

for $s > 0$, and $\alpha_{n,0} = 1$.

For $s = 0$ one can obtain an asymptotic formula for the quantity appearing in the theorem, and $0 < r < 1$, without relying on K. I. Babenko's results, but using the Cauchy integral formula and the estimate following from formula (6),

$$\int_{|z|=r} \left| \frac{1}{\xi - z} - Q_{n-1}(z, \xi) \right| \cdot |dz| < 2\pi r^{n+1} (1 + \varepsilon_n)$$

for any ξ , $|\xi| = 1$, where $Q_{n-1}(z, \xi)$ is some polynomial of degree $n - 1$ in z , $\varepsilon_n > 0$, and $\lim \varepsilon_n = 0$.

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Note: Figure translations are in progress. See original paper for figures.

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