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Abstract

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ON THE KINETICS OF WAVES IN A PLASMA

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1. It is known that kinetic processes in a weakly turbulent plasma are due to particle—particle, particle—wave, and wave—wave interactions. If the number of particles in the Debye sphere $N_D = (T/e^2 n^{1/3})^{3/2} \gg 1$ and the energy density of the oscillations is sufficiently small, then the influence of the last interaction may be neglected; in this case the principal role will be played by the particle—wave interaction ([1]). With increasing energy of the oscillations, the role of the wave—wave interaction grows and, under certain conditions, may exert a substantial influence on the kinetics of a weakly turbulent plasma.

The wave—wave interaction and some effects associated with it were considered in ([2–4]) on the basis of the kinetic equation for waves. In these works the kinetic equation for waves is derived from the hydrodynamic equations of the plasma. In the present work a derivation of the kinetic equations for waves is given by expanding the Lagrangian of a collisionless plasma in powers of the oscillation amplitude. From the kinetic equations obtained for interacting waves, some characteristic quantities of a weakly turbulent plasma are estimated.

2. Let us have a solution of the equation of motion of a particle of species α

$$m_\alpha \frac{\partial^2 \mathbf{r}^\alpha}{\partial t^2} = e_\alpha \mathbf{E}(\mathbf{r}^\alpha, t) + \frac{e_\alpha}{c}\left[\mathbf{v}^\alpha \mathbf{H}(\mathbf{r}^\alpha, t)\right] \tag{1}$$

with initial conditions: $\mathbf{r}^\alpha(\mathbf{r}_0^\alpha, \mathbf{v}_0^\alpha, 0) = \mathbf{r}_0^\alpha$, $\quad \mathbf{v}^\alpha(\mathbf{r}_0^\alpha, \mathbf{v}_0^\alpha, 0) = \mathbf{v}_0^\alpha$. Then the Lagrange function for the particle—field system will have the form

$$L = \sum_\alpha \iint d\mathbf{r}_0^\alpha d\mathbf{v}_0^\alpha f_\alpha(\mathbf{r}_0^\alpha, \mathbf{v}_0^\alpha) \left[\frac{1}{2} m_\alpha \left(\frac{\partial \mathbf{r}^\alpha}{\partial t}\right)^2 - e_\alpha V(\mathbf{r}^\alpha, t) + \frac{e_\alpha}{c}\frac{\partial \mathbf{r}^\alpha}{\partial t}\mathbf{A}(\mathbf{r}^\alpha, t)\right] +$$

$$+\frac{1}{8\pi}\int d\mathbf{r}\,(\mathbf{E}^2 - \mathbf{H}^2), \tag{2}$$

where

$$\mathbf{E} = -\vec{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{H} = \operatorname{rot} \mathbf{A}.$$

The summation is over particle species. The remaining notation is generally accepted.

Suppose that time-independent solutions of the field equations $\mathbf{E}_0, \mathbf{H}_0$ and the corresponding solution of the equation of motion (1) are known. These solutions describe a certain equilibrium state of the system. For the Lagrangian function of the perturbed state, after simple transformations from (3) we obtain ([5])

$$L = \sum_\alpha \iint d\mathbf{r} d\mathbf{v}\, f_\alpha(\mathbf{rv}) \left\{ \frac{1}{2} m_\alpha (\mathbf{v} + D^\alpha \mathbf{Y}^\alpha)^2 - e_\alpha V_0(\mathbf{r} + \mathbf{Y}^\alpha) - e_\alpha V'(\mathbf{r} + \mathbf{Y}^\alpha) + \right.$$

$$\left. + \frac{1}{c} e_\alpha (\mathbf{v} + D^\alpha \mathbf{Y}^\alpha) \left[\mathbf{A}_0(\mathbf{r} + \mathbf{Y}^\alpha) + \mathbf{A}'(\mathbf{r} + \mathbf{Y}^\alpha) \right] \right\} +$$

$$+ \frac{1}{8\pi} \int d\mathbf{r} \left[(\mathbf{E}_0 + \mathbf{E}')^2 - (\mathbf{H}_0 + \mathbf{H}')^2 \right], \qquad (3)$$

where $\mathbf{Y}^\alpha$ are the displacements of the particles, while V' and $\mathbf{A}'$ are the deviations of the potentials from the equilibrium values $V_0, \mathbf{A}_0$ due to the perturbation; D^α denotes the operator

$$D^\alpha = (\mathbf{v}\vec{\nabla}) + (\mathbf{a}^\alpha \vec{\nabla}_v) + \frac{\partial}{\partial t}, \qquad \mathbf{a}^\alpha = \frac{e_\alpha}{m_\alpha} \left(\mathbf{E}_0 + \frac{1}{c} [\mathbf{v}\mathbf{H}_0] \right).$$

Expanding the Lagrangian (3) in powers of $\mathbf{Y}^\alpha$, we write it in the form

$$L = \sum_n L_n.$$

L_0 contains only equilibrium quantities and therefore is of no interest. It can be shown that $L_1 \equiv 0$.

Using the expression for L_2, from the Euler–Lagrange equations we obtain

$$m_\alpha D^2 \mathbf{Y}^\alpha = e_\alpha \left(\mathbf{E}^1 + \frac{1}{c} [\mathbf{v}\mathbf{H}^1] \right) \frac{e_\alpha}{c} [D^\alpha \mathbf{Y}^\alpha \mathbf{H}_0] + e_\alpha \mathbf{Y}^\alpha \operatorname{div} \left(\mathbf{E}_0 + \frac{1}{c} [\mathbf{v}\mathbf{H}_0] \right); \quad (4)$$

$$\operatorname{div} \mathbf{E}^1 = -4\pi \sum_\alpha e_\alpha \int \operatorname{div}[\mathbf{Y}^\alpha f_\alpha(\mathbf{r}, \mathbf{v})]\, d\mathbf{v}; \qquad (5)$$

$$\operatorname{rot} \mathbf{H}^1 = \frac{1}{c} \frac{\partial \mathbf{E}^1}{\partial t} + \frac{4\pi}{c} \sum_\alpha e_\alpha \int \left\{ \frac{\partial \mathbf{Y}^\alpha}{\partial t} f_\alpha(\mathbf{r}, \mathbf{v}) + \operatorname{rot}[\mathbf{Y}^\alpha f_\alpha(\mathbf{r}, \mathbf{v}) \mathbf{v}] \right\} d\mathbf{v}. \qquad (6)$$

These equations contain the natural oscillations of the plasma.

3. Let us consider the interaction between longitudinal waves in an isotropic homogeneous plasma. In this case

$$L_3 = -\frac{1}{2}\sum_\alpha e_\alpha \iint d\mathbf{r}\, d\mathbf{v}\, f_\alpha(\mathbf{r},\mathbf{v}) Y_i^\alpha Y_k^\alpha \nabla_i \nabla_k V'(\mathbf{r}); \tag{7}$$

$$L_4 = -\frac{1}{6}\sum_\alpha e_\alpha \iint d\mathbf{r}\, d\mathbf{v}\, f_\alpha(\mathbf{r},\mathbf{v}) Y_i^\alpha Y_k^\alpha Y_l^\alpha \nabla_j \nabla_k \nabla_l V'(\mathbf{r}). \tag{8}$$

The displacements $\mathbf{Y}^\alpha$ are related to the field $-\vec{\nabla} V'(\mathbf{r})$ by the equation

$$m_\alpha D^2 \mathbf{Y}^\alpha = -e_\alpha \vec{\nabla} V'(\mathbf{r}). \tag{9}$$

In an isotropic homogeneous plasma there exist electron Langmuir (l-plasmons) and ion-acoustic (s-plasmons) oscillations (it is assumed that the condition for the existence of ion-acoustic waves, $T_e \gg T_i$, is satisfied). Representing the field potential in the form $V' = V'^l + V'^s$ and expanding V'^l and V'^s in a Fourier series,

$$V'^l = \frac{1}{\sqrt{\Omega}}\sum_k V_k'^l e^{i\mathbf{kr}}, \qquad V'^s = \frac{1}{\sqrt{\Omega}}\sum_q V_q'^s e^{i\mathbf{qr}},$$

from (9) we obtain:

$$\mathbf{Y}^\alpha = \frac{i}{\sqrt{\Omega}}\frac{e_\alpha}{m_\alpha}\left(\sum_k \mathbf{k}\frac{V_k'^l}{\omega_0^2} e^{i\mathbf{kr}} + \sum_q \mathbf{q}\frac{V_q'^s}{(\mathbf{qv})^2} e^{i\mathbf{qr}}\right), \qquad \omega_0 = \sqrt{\frac{4\pi e^2 n}{m}}. \tag{10}$$

L_3 describes three-plasmon processes: the decay of one wave into two and the fusion of two waves into one; L_4 describes four-plasmon processes: the decay of one wave into three waves, the fusion of three waves into one, and the scattering of one wave by another wave. In three-plasmon processes, by the conservation laws only the decay of an l-plasmon into an l-plasmon and an s-plasmon, and the reverse process, is allowed. Taking this into account, substituting (10) into (7), and retaining the principal terms, we obtain

$$L_3 = \frac{1}{\sqrt{\Omega}}\sum_{k+q+k_1=0} \Phi_{kk_1} V_k'^l V_q'^s V_{k_1}'^l, \tag{11}$$

where

$$\Phi_{kk_1} = \frac{1}{2}\left(\sum_\alpha \frac{e_\alpha^3 n_\alpha}{m_\alpha^2 \omega_0^2}\left\langle \frac{1}{v_\alpha^2}\right\rangle\right)(\mathbf{kk}_1). \tag{12}$$

Using the classical Lagrangian (11), on the basis of the method of secondary quantization, the kinetic equations that determine the change of the distribution function of l-plasmons N_k^l and s-plasmons N_q^s per unit time as a result of the processes described by the Lagrangian L_3 are constructed as follows:

in the usual way. These equations have the form

$$\frac{\partial N_k^l}{\partial t} = \mathrm{St}^{(3)}(N_k^l) = \sum_{q,k_1} W_{N_k^l N_q^s N_{k_1}^l}^{N_{k+1}^l, N_{q+1}^s, N_{k_1-1}^l} \left[(N_k^l+1)(N_q^s+1)N_{k_1}^l - N_k^l N_q^s (N_{k_1}^l+1)\right] \delta(\omega_{k_1}^l - \omega_k^l - \omega_q^s) +$$

$$+ \sum_{q,k_1} W_{N_k^l N_q^s N_{k_1}^l}^{N_{k-1}^l, N_{q+1}^s, N_{k_1+1}^l} \left[(N_k^l+1)N_{k_1}^l N_q^s - N_k^l (N_{k_1}^l+1)(N_q^s+1)\right] \times$$

$$\times \delta(\omega_k^l - \omega_{k_1}^l - \omega_q^s); \qquad (13)$$

$$\frac{\partial N_q^s}{\partial t} = \mathrm{St}^{(3)}(N_q^s) = 2\sum_{k,k_1} W_{N_k^l N_q^s N_{k_1}^l}^{N_{k+1}^l N_{q+1}^s, N_{k_1-1}^l} \times$$

$$\times \left[(N_q^s+1)(N_k^l+1)N_{k_1}^l - N_q^s N_k^l (N_{k_1}^l+1)\right] \delta(\omega_{k_1}^l - \omega_k^l - \omega_q^s). \qquad (14)$$

In these equations

$$W_i^f = \frac{2\pi}{\hbar^2} |\langle f|L_3|i\rangle|^2 = \frac{1}{\Omega}\frac{2\pi}{\hbar^2} |\Phi_{kk_1}|^2 |V_k'^l|^2 |V_{k_1}'^l|^2 |V_q'^s|^2, \qquad (15)$$

where

$$V_k'^l = \frac{1}{k}\sqrt{2\pi\hbar\omega_k^l}, \qquad V_q'^s = \frac{1}{q}\sqrt{2\pi\hbar\omega_q^s}. \qquad (16)$$

The kinetics of three-plasmon processes in a weakly turbulent plasma in the “random phases” approximation, when the interaction between waves reduces to their collisions with one another, is completely described by equations (13) and (14).

Let us turn to the kinetics of four-plasmon processes. The decay of an l-plasmon into three plasmons and the fusion of three into one l-plasmon are forbidden by the conservation laws. Therefore the change in the distribution function of l-waves is due to the process of scattering of an l-plasmon by an l-plasmon and to processes described by the Lagrangian L_3 in second order of perturbation theory. The kinetic equation describing this change has the form

$$\frac{\partial N_k^l}{\partial t} = \mathrm{St}^{(4)}(N_k^l) + \mathrm{St}(N_k^l). \qquad (17)$$

For the collision integrals $\mathrm{St}^{(4)}(N_k^l)$, $\mathrm{St}(N_k^l)$, according to (8), (10), (11), and (12), we have:

$$\mathrm{St}^{(4)}(N_k^l) = 2\sum_{k_1k_2k_3} W_{N_k^l N_{k_1}^l N_{k_2}^l N_{k_3}^l}^{N_{k+1}^l, N_{k_1+1}^l, N_{k_2-1}^l, N_{k_3-1}^l} \Big[(N_k^l+1)(N_{k_1}^l+1)N_{k_2}^l N_{k_3}^l -$$
$$- N_k^l N_{k_1}^l (N_{k_2}^l+1)(N_{k_3}^l+1)\Big]\,\delta(\omega_k^l+\omega_{k_1}^l-\omega_{k_2}^l-\omega_{k_3}^l) +$$
$$+\sum_{k_1k_2k_3} W_{N_k^l N_{k_1}^l N_{k_2}^l N_{k_3}^l}^{N_{k+1}^l, N_{k_1-1}^l, N_{k_2-1}^l, N_{k_3+1}^l} \Big[(N_k^l+1)(N_{k_3}^l+1)N_{k_1}^l N_{k_2}^l -$$
$$- N_k^l N_{k_3}^l (N_{k_1}^l+1)(N_{k_2}^l+1)\Big]\,\delta(\omega_k^l+\omega_{k_3}^l-\omega_{k_1}^l-\omega_{k_2}^l); \qquad (18)$$

$$\mathrm{St}(N_k^l) = 2\sum_{qq'k'} B_{kqq'k'}\Big[(N_k^l+1)(N_q^s+1)N_{q'}^s N_{k'}^l -$$
$$- N_k^l N_q^s (N_{q'}^s+1)(N_{k'}^l+1)\Big]\,\delta(\omega_k^l+\omega_q^s-\omega_{k'}^l-\omega_{q'}^s), \qquad (19)$$

where

$$W_i^f = \frac{2\pi}{\hbar^2}\,|\langle f|L_4|i\rangle|^2 = \frac{1}{\Omega^2}\frac{2\pi}{\hbar^2}\,|\psi_{kk_1k_2k_3}|^2|V_k'^l|^2|V_{k_1}'^l|^2|V_{k_2}'^l|^2|V_{k_3}'^l|^2; \qquad (20)$$

$$\psi_{kk_1k_2k_3} = \frac{1}{6}\left(\sum_\alpha \frac{n_\alpha e_\alpha^4}{m_\alpha^3\omega_0^6}\right)(\mathbf{k}\mathbf{k}_3)(\mathbf{k}_1\mathbf{k}_3)(\mathbf{k}_2\mathbf{k}_3); \qquad (21)$$

$$B_{kqq'k'} = \frac{(2\pi)^7\hbar^2}{\Omega^2}\,\frac{\omega_k^l\omega_q^s(\omega_{k+q}^l)^2\omega_{k'}^l\omega_{q'}^s}{(\omega_k^l+\omega_q^s-\omega_{k+q}^l)^2}\,\frac{1}{k^2q^2(k+q)^2{q'}^2{k'}^2}\,\left|\Phi_{\mathbf{k},\mathbf{k}+\mathbf{q}}\right|^2\left|\Phi_{\mathbf{k}+\mathbf{q},\mathbf{k}'}\right|^2. \qquad (22)$$

The kinetic equation for s-waves is more cumbersome than equation (17), since it contains terms due to the processes of the splitting of one wave into three and the merging of three waves into one. The structure of these terms is clear, and therefore we do not present this equation here.

The equations of wave kinetics in a weakly turbulent plasma (13), (14), (17) have the same form as the equations for the distribution functions of elementary excitations in condensed media ([6–10]). With the aid of equations (13), (14), and (17), some characteristic quantities of a weakly turbulent plasma can be estimated.

Let ion-acoustic waves propagate in a plasma in which electron Langmuir oscillations are excited. We shall determine the change in the number of s-waves due to interaction with l-oscillations. We note that this problem is analogous to the problem of sound absorption by thermal phonons in a solid, which was considered in ([11]). Taking into account that in the sound wave $N_q^s \gg 1$ and substituting $N_k^l = E_k^2/\hbar\omega_0$, from (14) it is not difficult to find the order of the collision time of s-plasmons with l-plasmons:

$$\tau_3 \sim \frac{1}{\omega_q^s}\left(\frac{nT}{E^2}\right), \qquad E^2 = \int E_k^2\,dk.$$

For the relaxation time of l-waves due to 4-plasmon processes, from (17), (18), and (19) we find:

$$\tau_4 \sim \frac{1}{\omega_0}\left(\frac{nT}{E^2}\right)^2.$$

The time of nonlinear damping of a packet of l-waves

$$\int_{(\Delta k)} E_k^2\,dk = E_{k_0}^2 \Delta k_0$$

is

$$\tau \sim \tau_4 \sim \frac{1}{\omega_0}\left(\frac{nT}{E_{k_0}^2 \Delta k_0}\right)^2.$$

If one introduces the "effective temperature" of the gas of l-plasmons by means of the equality $E^2 = T^l$, then for the coefficient of thermal conductivity in the gas of l-waves we shall have:

$$\varkappa \sim C_T R_D \left(\frac{nT}{E^2}\right)^2.$$

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Note: Figure translations are in progress. See original paper for figures.