FREQUENCY CONDITIONS FOR ABSOLUTE STABILITY OF CONTROL SYSTEMS WITH HYSTERESIS NONLINEARITIES

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Abstract

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FREQUENCY CONDITIONS FOR ABSOLUTE STABILITY OF CONTROL SYSTEMS WITH HYSTERESIS NONLINEARITIES

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Below, unless otherwise stated, we denote by capital Latin letters $v \times v$ matrices, by lowercase Latin letters $v \times 1$ column vectors (exception: t is time), and by Greek letters scalar quantities. Matrices, vectors, and numbers are real; I is the identity matrix.

1°. We shall study systems (see (1–5), etc.)

$$dz/dt = Pz + q\varphi[\sigma, \varphi_0]_t, \qquad \sigma = (z, r), \tag{1}$$

where $\varphi[\sigma, \varphi_0]_t$ is a hysteresis or relay-hysteresis function (see Fig. 1, where these "functions" are shown for $\sigma \geq 0$). Let us give an exact definition of a hysteresis function.

Definition 1. Denote by $\mathfrak{M}_{\sigma_0}(t_0, t_1)$ the set of functions $\sigma(t)$, continuous on $[t_0, t_1]$, such that $\sigma(t_0) = \sigma_0$. Suppose that: 1) to each σ_0 there is assigned some set $\mathfrak{E}(\sigma_0)$ of "initial values of the hysteresis function" (see Fig. 1); 2) for any $t_1 \geq t_0 \geq 0$ and $\varphi_0 \in \mathfrak{E}(\sigma_0)$ an operator $\varphi[\,\cdot\,|_{t_0}^{t_1}, \varphi_0]$ is specified, mapping $\mathfrak{M}_{\sigma_0}(t_0, t_1)$ into $C(t_0, t_1)$; 3) the relations $\varphi[\sigma|_{t_0}^{t_1}, \varphi_0]_{t_0} = \varphi_0$, $\varphi[\sigma|_{t_0}^{t_1}, \varphi_0]_t \in \mathfrak{E}[\sigma(t)]$ hold, where $\varphi[\sigma|_{t_0}^{t_1}, \varphi_0]$ is the corresponding function and $\varphi[\sigma|_{t_0}^{t_1}, \varphi_0]_t$ is its value at the point t, $t_0 \leq t \leq t_1$; 4) if $\sigma(t) = \sigma_1(t) \in \mathfrak{M}_{\sigma_0}(t_0, t_*)$ for $t_0 \leq t \leq t_*$ and $\sigma(t) = \sigma_2(t) \in \mathfrak{M}_{\sigma_*}(t_*, t_1)$ for $t_* \leq t \leq t_1$, where $\sigma_* = \sigma(t_*)$, then $\varphi[\sigma|_{t_0}^{t_1}, \varphi_0]_t = \varphi[\sigma_1|_{t_0}^{t_*}, \varphi_0]_t$ for $t_0 \leq t \leq t_*$ and $\varphi[\sigma|_{t_0}^{t_1}, \varphi_0]_t = \varphi[\sigma_2|_{t_*}^{t_1}, \varphi_*]_t$ for $t_* \leq t \leq t_1$, where $\varphi_* = \varphi[\sigma_1|_{t_0}^{t_*}, \varphi_0]_{t_*}$. When these conditions are fulfilled, the family of mappings $\varphi[\,\cdot\,|_{t_0}^{t_1}, \varphi_0]$ will be called a **continuous hysteresis function**. If all these mappings are continuous, then the hysteresis function is called **strongly continuous**.

Other definitions close in character are also possible, for example those obtained by replacing $C(t_0, t_1)$ with $L(t_0, t_1)$. If $\varphi[\sigma|_{t_0}^{t_1}, \varphi_0] \notin C(t_0, t_1)$, then the hysteresis function is called discontinuous (or relay-hysteresis). In Fig. 1 are depicted (with the usual interpretation understood) a strongly continuous (a), a continuous (b), and a discontinuous (c) hysteresis function*.

Fig. 1

Figure 1: Fig. 1

Fig. 1

In the present note we shall consider families of operators satisfying only conditions 1), 2) with fixed $t_0 = 0$; in this case we shall denote $\varphi[\sigma, \varphi_0] = \varphi[\sigma|_0^t, \varphi_0]$. In the case when $\varphi[\,\cdot\,, \varphi_0]$ is conti- discontinuous operator (in particular, a strongly discontinuous hysteresis function), the local existence theorem for equations (1) is proved in the usual way by applying Schauder' s principle*. If $\varphi[\cdot, \varphi_0]$ is a discontinuous operator and, still more, if $\varphi[\cdot, \varphi_0]$ is a relay-hysteresis function, the proof of the existence theorem remains an unsolved mathematical problem. For relay-hysteresis functions, the very concept of a solution of system (1) requires definition; this definition may be analogous to the definition of a solution of systems of differential equations with discontinuous right-hand sides ($^{6-9}$). The subsequent exposition is valid for any definition of a solution under which: 1) the solution $z(t)$ is an absolutely continuous function; 2) in equation (1), $\varphi[\sigma, \varphi_0]_t = \varphi(t)$ is an arbitrary function, L-integrable on any finite interval, connected with $\sigma(t)$ by relations formulated below; 3) equation (1) is satisfied almost everywhere; 4) if a solution exists and is bounded on $[0, \tau)$, then it can be continued to the interval $[\tau, \tau + \varepsilon)$, $\varepsilon > 0$.

2°. We shall, as usual, assume that for any $\varphi_0 \in \mathfrak{E}(\sigma_0)$ and any $t \geqslant 0$ from the existence interval, the following is fulfilled**:

$$0 \leqslant \sigma(t)\varphi[\sigma, \varphi_0]_t \leqslant \mu_0 \sigma^2(t), \qquad \varphi[\sigma, \varphi_0]_t = 0 \quad \text{when } \sigma(t) = 0. \tag{2}$$

Definition 2. System (1) is called **absolutely stable** if: 1) every solution of system (1) exists on $[0, \infty)$ and $z(t) \to 0$ as $t \to \infty$, and 2) there exists a continuous increasing function $\psi(\rho)$, depending only on P, q, r, μ_0, defined on $[0, \infty)$, $\psi(0) = 0$, such that for any $\varphi_0 \in \mathfrak{E}(\sigma_0)$, where $\sigma_0 = (z(0), r)$, the inequality $|z(t)| \leqslant \psi(|z(0)|)$ is fulfilled.

Denote

$$\chi(\lambda) = ((P - \lambda I)^{-1} q, r), \qquad \xi(\omega) = 1/\mu_0 + \operatorname{Re}\chi(i\omega), \qquad \eta(\omega) = \omega \operatorname{Im}\chi(i\omega).$$

For absolute stability it is necessary that the curve $\xi(\omega) + i\eta(\omega)$, $0 \leqslant \omega < \infty$, not intersect the negative real semiaxis***.

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* In case b the mapping is discontinuous, for example, on functions $\sigma(t)$ having a maximum at the point σ_2, provided that $\varphi_0 = \varphi_0'$.

Theorem 1. *Suppose that the spectrum of the matrix P lies in the open left half-plane, $\xi(\omega) > 0$ for $0 \leqslant \omega < \infty$, and $+\infty \geqslant \lim_{\omega\to\infty} \omega^2 \xi(\omega) > 0$. Then system (1) is absolutely stable and, moreover, there exist $\varkappa_0, \varkappa > 0$, depending only on P, q, r, μ_0, such that*

$$|z(t)| \leqslant \varkappa_0 e^{-\varkappa t} |z(0)|. \tag{3}$$

* Of course, the same is true for equations $\dot{x} = f[x, f_0]$, where $f[x, f_0] = f[x_0^t, f_0]_t$ is an analogously defined family of operators on vector functions $x(t')$, $0 \leqslant t' \leqslant t$.

** This condition is obviously fulfilled (with the corresponding completion of the definition for values $\sigma < 0$) for the hysteresis functions depicted in Fig. 1. Note that the case $\mu_0 = \infty$ is allowed; then, in subsequent formulas, $1/\mu_0 = 0$.

*** It is known from linear theory that this condition is necessary and sufficient for the stability of all linear systems (1) with $\varphi[\sigma, \varphi_0] = \mu\sigma$, $0 \leqslant \mu \leqslant \mu_0$. The amplitude-phase characteristic of the linear part of the system $[-\chi(i\omega)]$ can, as is well known, be obtained experimentally.

Let us note that the conditions of the theorem are satisfied by systems (1) with hysteresis nonlinearities (of the type shown in Fig. 1), whose graphs may vary with time while remaining in the fixed sector $0 \leq \sigma\varphi \leq \mu_0 \sigma^2$.

Definition 3. The operator $\varphi[\cdot, \varphi_0]$ is called a (+)-operator if relations (2) are satisfied and, for any absolutely continuous function $\sigma(t)$ and $\varphi_0 \in \mathfrak{E}[\sigma(0)]$, there exists the integral

$$\Phi(t) \int_0^t \varphi[\sigma, \varphi_0]_t \, d\sigma(t)$$

and

$$\Phi(t) \geq -\psi_0(|\sigma(0)|),$$

where $\psi_0(\sigma)$ is a continuous, nondecreasing function, $\psi_0(0) = 0$.

Speaking somewhat imprecisely, a hysteresis function is a (+)-operator if the increment of area along any of its hysteresis loops is positive. If, on the other hand, the increment of area along any hysteresis loop of the function $\varphi[\sigma, \varphi_0]$ is negative, then the hysteresis function $\mu_0\sigma - \varphi[\sigma, \varphi_0]$ is a (+)-operator.

The hysteresis functions shown in Fig. 1 and extended in the corresponding way to negative values of σ are (+)-operators (it is assumed that their graphs do not vary with time). If, in Fig. 1, b, c, the directions of the arrows are changed, then the operators $\mu_0\sigma - \varphi[\sigma, \varphi_0]$ will be (+)-operators. It is also easy to give examples of strongly discontinuous hysteresis functions $\varphi[\sigma, \varphi_0]$ such that $\mu_0\sigma - \varphi[\sigma, \varphi_0]$ are (+)-operators.

Theorem 2. Suppose that the spectrum of the matrix P lies in the open left half-plane, and also either a′) $\varphi[\cdot, \varphi_0]$ is a (+)-operator and b′) the curve $\xi(\omega) + i\eta(\omega)$ is located, for $0 \leq \omega \leq \infty$, in some open half-plane bounded by a straight line passing through the origin and containing neither the negative real nor the positive imaginary semiaxis; or a″) $\mu_0 \neq \infty$, the operator $\mu_0\sigma - \varphi[\sigma, \varphi_0]$ is a (+)-operator, and b″) condition b′) is satisfied with the words "positive imaginary" replaced by "negative imaginary." Then system (1) is absolutely stable.

Theorem 3. Suppose that: 1) the matrix P has one zero eigenvalue and the remaining ones are in the open left half-plane; 2) $\varphi[\sigma, \varphi_0]_t \neq 0$ when $\sigma(t) \neq 0$; and 3) $\Gamma^2 = \mathrm{Res}_{\lambda=0}\, \chi(\lambda) > 0$*. For absolute stability of system (1) it is sufficient that either $\xi(\omega) > 0$ for $0 \leq \omega \leq \infty$, or conditions a′), b′), or conditions a″), b″) of Theorem 2 be satisfied for $0 \leq \omega \leq \infty$.

Remark. Under the assumptions of the theorem,

$$\xi(0) = \chi_0(0),$$

where

$$\chi_0(\lambda) = \chi(\lambda) - \Gamma^2/\lambda, \qquad \eta(0) = -\Gamma^2, \qquad \xi(\infty) = 1/\mu_0, \qquad \eta(\infty) = (q, r).$$

Theorem 4. Suppose that: 1) the matrix P has one zero, two purely imaginary eigenvalues $\pm i\omega_0$, and its remaining eigenvalues lie in the open left half-plane; 2) $\varphi[\sigma, \varphi_0]_t \neq 0$ when $\sigma(t) \neq 0$. Represent the function $\chi(\lambda)$ in the form

$$\chi(\lambda) = \Gamma^2/\lambda + (\alpha\lambda + \beta)/(\lambda^2 + \omega_0^2) + \chi_1(\lambda),$$

where $\chi_1(\lambda)$ is a function holomorphic on the imaginary axis. For absolute stability of system (1) it is sufficient that

$$\Gamma^2 > 0, \qquad \alpha \geq 0, \qquad -\beta(q, r) + \alpha\omega_0^2/\mu_0 > 0,$$

$$\beta(\Gamma^2 - \alpha) + \alpha\omega_0^2/\mu_0 + \mathrm{Re}[(\alpha\omega_0^2 + i\omega\beta)\chi_1(i\omega)] > 0, \qquad 0 \leq \omega < \infty, \qquad (4)$$

and that, for $\beta > 0$, condition a′) be satisfied, while for $\beta < 0$ condition a″) of Theorem 2 be satisfied.

3°. In the case where $\varphi[\sigma, \varphi_0] = \varphi(\sigma)$ is an ordinary function (then, evidently, $\varphi[\sigma, \varphi_0]$ and $\mu_0\sigma - \varphi[\sigma, \varphi_0]$ are (+)-operators), the assertions of the theo-

Theorems 2 and 3 are known—they coincide with the criterion of V. M. Popov (^{10}, ^{11}).* Theorem 4 is new also in this case; Theorems 1-4 give a new result also in the case when the nonlinearity is an ordinary function depending on t.

The proof of Theorems 1-4 differs substantially from (^{10}, ^{11}) and is based on the results of (^{12}). To illustrate the method we give a very simple proof of Theorem

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* The condition $\Gamma^2 > 0$ is the well-known necessary stability condition of A. I. Lur' e [1].

1 (unlike the proofs of Theorems 2-4). For $\Omega = (Hz, z)$ we have, by virtue of system (1),

$$-\dot{\Omega} = (Gz, z) + 2\varphi(g, z) + \sigma\varphi,$$

where

$$-G = P^*H + HP, \qquad g = -Hq - r/2.$$

We require that

$$G - \mu_0 gg^* > 0 \quad \text{for } \mu_0 \neq \infty,$$

and

$$G > 0, \quad g = 0 \quad \text{for } \mu_0 = \infty.$$

According to (^{12}), the matrix $H = H^* > 0$ exists if the conditions of the theorem are satisfied. In this case

$$\dot{\Omega} < -2\varkappa\Omega$$

for some $\varkappa > 0$. Consequently, the solution is continuable on $(0, \infty)$, and estimate (3) holds.

The sufficient conditions of Theorems 1-4 are, in a certain sense, optimal. Thus, for example, in Theorem 4 the inequalities $\Gamma^2 > 0$, $\alpha \geq 0$ are necessary for absolute stability, and the inequalities (4) with the signs $\geq$ are necessary in order that absolute stability could be detected by means of a Lyapunov function of the form

$$\Omega = (Hz, z) + \vartheta \int_0^t \varphi[\sigma, \varphi_0]_\tau \, d\sigma(t)$$

with derivative $\dot{\Omega} \leq 0$. The inequalities (4) are necessary and sufficient for the derivative $\dot{\Omega}$ to be minimally degenerate. The function

$$\dot{\Omega} = \Omega_0(z, \varphi) \leq 0$$

is called minimally degenerate if

$$\text{Inf}[-\Omega_0(0, \varphi)/\varphi^2] > 0$$

and from $\Omega_0(z, \varphi) = 0$ it follows that $\varphi = 0$, while z lies in the sum $\mathfrak{Q}$ of the eigensubspaces of the matrix P corresponding to zero and purely imaginary eigenvalues. It can be proved that, for any function Ω of the indicated form such that $\Omega_0(z, \varphi) \leq 0$, from $\Omega_0(z, 0) = 0$ it follows that $z \in \mathfrak{Q}$, which explains the definition of minimal degeneracy.

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* Theorems 1 and 2 (^{12}) give, in essence, a new proof of this Popov criterion. Suppose, for simplicity, that the spectrum of P lies in the open left half-plane. For

$$\Omega = (Hz, z) + \vartheta \int_0^\sigma \varphi(\sigma)\, d\sigma$$

we have

$$-\dot{\Omega}(Gz, z) + 2\varphi(g, z) + \gamma\varphi^2 + \varphi(\sigma - \varphi/\mu_0),$$

where G, g, γ are expressed in terms of P, q, r, ϑ, H. Requiring that either

$$\gamma > 0, \quad \gamma G - gg^* > 0,$$

or

$$\gamma = 0, \quad g = 0, \quad G > 0,$$

be fulfilled, we obtain

$$\Omega > 0, \qquad \dot{\Omega} < 0 \quad \text{for } |z| \neq 0,$$

whence absolute stability follows. By Theorems 1 and 2 (^{12}), a solution H of the inequalities obtained exists if

$$\xi(\omega) + \vartheta\eta(\omega) > 0, \qquad +\infty > \lim_{\omega\to\infty} \omega^2[\xi(\omega) + \vartheta\eta(\omega)] > 0,$$

which is what was required to prove.

Note: Figure translations are in progress. See original paper for figures.

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