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Abstract

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STABILITY THEOREMS FOR CONVEX SURFACES CLOSE TO A SPHERE

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1. To every uniqueness theorem in differential geometry there corresponds a certain stability theorem, asserting that if the conditions are satisfied with a prescribed error ε, then the conclusion is also true with accuracy up to $\varphi(\varepsilon)$. Only in a few cases is the function $\varphi(\varepsilon)$ known or at least estimated. We wish to prove a stability theorem corresponding to Liebmann's theorem on the rigidity of the sphere, generalized to the n-dimensional case. It is essential that, in this, the order of our estimate $\varphi(\varepsilon)$ does not depend on the dimension n.

In this note $C_1, C_2, \ldots$ denote numbers depending only on n.

2. We first prove a stability theorem corresponding to the known theorem (([1], p. 141; we were unable to consult the works cited in ([1])): a convex body all of whose projections are balls (not necessarily of equal radius) is a ball.

Let T^n be a convex body in Euclidean space E^n, and let $R(T^n)$, $r(T^n)$ be the radii of the smallest circumscribed and, respectively, largest inscribed ball of T^n. Put

$$\delta(T^n) = R(T^n) - r(T^n).$$

Denote by $P^k(T^n)$ the orthogonal projection of T^n onto an arbitrary k-dimensional plane $P^k \subset E^n$.

Theorem 1. *If for all planes P^k of a given dimension k $(2 \leq k \leq n)$*

$$\delta(P^k(T^n)) < \varepsilon,$$

then

$$\delta(T^n) < C\varepsilon,$$

where C is a constant depending only on n.

Lemma 1. *Let S_1^p, S_2^p be spheres of radii r_1, r_2 in E^{p+1}, $0 < C_1 \leq r_1 \leq r_2 \leq C_2$, and let points $a_1, b_1 \in S_1^p$, $a_2, b_2 \in S_2^p$, with distance $\rho(a_1, b_1) \geq C_3 > 0$, $\rho(a_1, a_2) < \varepsilon$, $\rho(b_1, b_2) < \varepsilon$. If S_1^p lies inside the sphere S_3^p of radius $r_1 + \varepsilon$, concentric with S_2^p, then there exists C_4 such that*

$$|r_1 - r_2| < C_4\varepsilon.$$

Drawing a plane $E^3 \subset E^{p+1}$ containing a_1, b_1 and the centers of S_1^p, S_2^p, one easily reduces the proof of the lemma to the case $p = 2$.

Proof of Theorem 1. It is enough to consider the case $k = n-1$, since the general case is then proved by induction. Further, if ρ is the minimal value of $R(P^{n-1}(T^n))$ over all P^{n-1}, then T^n is contained in a cylinder of radius ρ, and for any other P_1^{n-1} the width of the body $P_1^{n-1}(T^n)$ is no greater than 2ρ. Therefore one may assume that every P^{n-1} contains a ball $K^{n-1}(P^{n-1})$ of unit radius for which the Fréchet distance between the surfaces of the bodies is

$$\rho(K^{n-1}(P^{n-1}),\, P^{n-1}(T^n)) < \varepsilon.$$

Construct the smallest ball K^n containing T^n; by Jung' s known theorem, the radius of K^n does not exceed a certain C_2. Further, there is a pair of contact points a, b of the body T^n with the boundary sphere S^{n-1} of the ball K^n, the distance

between which is not less than $\sqrt{2}/2$; indeed, otherwise by a suitable translation one could obtain from K^n a ball containing T^n strictly inside.

Construct some plane P_0^{n-1} containing a, b and the center of K^n, and put $P_0^{n-1}(K^n) = K^{n-1}$; obviously, the radii of K^n and K^{n-1} are equal, $P_0^{n-1}(T^n) \subset K^{n-1}$, and the projections a', b' of the points a, b onto P_0^{n-1} are points of tangency of the boundaries of $P_0^{n-1}(T^n)$, K^{n-1}. Further,

$$\rho(K^{n-1}(P_0^{n-1}), P_0^{n-1}(T^n)) < \varepsilon,$$

so that

$$K^{n-1}(P_0^{n-1}) \subset K_\varepsilon^{n-1},$$

where K_ε^{n-1} is a ball, concentric with K^{n-1}, of radius larger by ε. On the other hand, a' and b' are at distance from the boundary of $K^{n-1}(P_0^{n-1})$ not exceeding ε and belong to the boundary of K^{n-1}; by Lemma 1, the radius of K^n is less than $1 + C_4\varepsilon$. Having constructed the largest ball $K^{n'}$ contained in T^n and applying Lemma 1 again, we similarly find that the radius of $K^{n'}$ is greater than $1 - C_5\varepsilon$, which proves the theorem.

3. Let us introduce a class of symmetrizations for convex surfaces in E^n, containing, as particular cases, the known symmetrizations of Steiner and Schwarz. Let E^p be a plane in E^n, and $T^n \subset E^n$ a convex body. For any point $x \in E^p$ construct the ball K_x^{n-p} with center at x, lying in the orthogonal $(n-p)$-dimensional plane E_x^{n-p} and having volume equal to the volume of $T^n \cap E_x^{n-p}$. Put

   $$S_{n,p}(T^n) = \bigcup_{x \in E^p}^{p} K_x^{n-p};$$

   $S_{n,1}$ is the Schwarz symmetrization, $S_{n,n-1} = S$ is the Steiner symmetrization.

It is easy to see that every symmetrization $S_{n,p}$ reduces to a finite number of Steiner symmetrizations with respect to suitable $(n-1)$-dimensional planes, followed by a passage to the limit. Hence, in particular, it follows that $S_{n,p}$ preserves volume, does not increase diameter, and that Theorem 4 (^{2}) remains valid when S is replaced by $S_{n,p}$.

4. The following theorem can be used to prove a number of stability theorems in which the surfaces turn out to differ little from a sphere.

Theorem 2. *Let Φ be a class of convex surfaces in E^n possessing the following properties:*

1) The diameter of a surface of the class Φ is not greater than $2+\varepsilon$, and its width is less than $2-\varepsilon$.

2) The class Φ, together with every surface F^{n-1}, contains the surface $S(F^{n-1})$ obtained by Steiner symmetrization.

3) The class Φ, together with a sequence of surfaces F_i^{n-1}, contains their limit surface.

Then every surface of the class Φ is contained in a ball of radius $1+C\sqrt{\varepsilon}$ and contains a ball of radius $1-C\sqrt{\varepsilon}$, where C depends only on n.

Proof. First we prove

Lemma 2. *The orthogonal projection of a surface F^{n-1} onto an arbitrary two-dimensional plane P^2 has area not less than $\Pi-\varepsilon$. Let the area of $P^2(F^{n-1})$ be equal to σ and $\Pi\tau^2=\sigma$.*

Construct the plane P^{n-2}, orthogonal to P^2, and perform the symmetrization $S_{n,n-2}$ with respect to P^{n-2}; according to item 3,

$$F_1^{n-1}=S_{n,n-2}(F^{n-1})\in\Phi.$$

It is clear that all sections of F_1^{n-1} parallel to P^2 have area not greater than σ; therefore the radii of all K_x^{n-2} are not greater than τ. Thus, the distances of all points of F_1^{n-1} from P^{n-2} do not exceed τ. If now one constructs some plane $P^{n-1}\supset P^{n-2}$ and two parallel P^{n-1}-planes P_1^{n-1}, P_2^{n-1} on different sides and at distance τ from P^{n-1}, then F_1^{n-1} lies between P_1^{n-1}, P_2^{n-1}. But then the width

$$\Delta(F_1^{n-1})\le 2\tau;$$

on the other hand, $F_1^{n-1}\in\Phi$, consequently,

$\Delta(F_1^{n-1})\geqslant 2-\varepsilon$, so that $\tau\geqslant 1-\varepsilon/2$, whence the assertion of the lemma follows.

To prove Theorem 2, let us note that for any $F^{n-1}\in\Phi$ and any P^2 the width of $P^2(F^{n-1})$ in any direction does not exceed $2+\varepsilon$; hence, as is known, it follows that the length of the perimeter of the domain $P^2(F^{n-1})$ satisfies $L\leqslant\pi(2+\varepsilon)$. By Lemma 2, the area of $P^2(F^{n-1})$ is $F\geqslant\pi-\varepsilon$. From Bonnesen' s inequality ((1), p. 83) it follows that

$$4\pi(P-\rho)^2 \leqslant L^2 - 4\pi F,$$

where P, ρ are the radii of concentric circles respectively containing $P^2(F^{n-1})$ and contained in $P^2(F^{n-1})$. Thus $P-\rho < C_6\sqrt{\varepsilon}$; from Lemma 2 we find that $P \geqslant 1-\varepsilon/2$, and it remains to apply Theorem 1. Let us note that, for small ε, one may take for C any number greater than $\sqrt{8\pi}$.

5. From Theorem 2, in particular, one can derive

Theorem 3. *Let the Gaussian curvature of a convex surface F^{n-1} be between $1-\varepsilon$ and $1+\varepsilon$. Then F^{n-1} is contained in a ball of radius $1+C\sqrt{\varepsilon}$ and contains a ball of radius $1-C\sqrt{\varepsilon}$, where C depends only on n.*

Proof. Consider the class Φ_1 of convex surfaces satisfying the following conditions:

1. The diameter of the surfaces Φ_1 is not greater than $2+C_7\varepsilon$.
2. The volume of the surfaces Φ_1 is not less than $\chi_n - C_7\varepsilon$, where χ_n is the volume of the unit n-dimensional ball.
3. The Gaussian curvature of the surfaces Φ_1 is not less than $1-C_7\varepsilon$.

To prove Theorem 3 it suffices to verify two assertions:

a) the class Φ_1 satisfies the conditions of Theorem 2 with some $C_8\varepsilon$ in place of ε;

b) every surface satisfying the conditions of Theorem 3 belongs to the class Φ_1 for some C_7.

The proof of a) and b) is based on results of V. I. Diskant (^{3}). From Theorem 3 and the remark in item (4) (^{3}) it follows that the width $\Delta(F^{n-1}) \geqslant 2-C_9\varepsilon$ for all $F^{n-1} \in \Phi_1$, so that condition 1) of our Theorem 2 is satisfied with $\max(C_7, C_9)$ in place of ε. From Theorem 4 (^{2}) it follows that conditions 2), 3) are also satisfied, and (a) is proved. b) follows from Theorem 2 (^{3}) and Theorem 3 (^{2}).

I consider it my duty to express my gratitude to V. A. Toponogov, who noted that the theorems stated in (^{2}) may be useful in the question of stability.

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3. V. I. Diskant, DAN, 153, No. 3 (1963).

Note: Figure translations are in progress. See original paper for figures.