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Abstract

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ON THE EQUATION $\frac{d\varphi}{dt} = f - \alpha$ ON THE PHASE SPACE OF A DYNAMICAL SYSTEM

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Let $\{S_t\}$ be a dynamical system, or measurable flow, in a Lebesgue space R with normalized measure m. We shall assume that the measure m is indecomposable for the system $\{S_t\}$. Denote by $L^2_m(R)$ the space of all complex-valued functions defined on R with square-integrable modulus with respect to the measure m, and consider the equation

$$\varphi(S_t x) = \varphi(x) + \int_0^t [f(S_\tau x) - \alpha]\, d\tau, \tag{1}$$

where $f(x)$ is a given function from $L^2_m(R)$, and α is its mean value, i.e.

$$\alpha = \int_R f(x)\, dm,$$

while $\varphi(x)$ is the required function, also from $L^2_m(R)$. Here (1) must hold for any t at the points of some set $M \subset R$, invariant with respect to the system $\{S_t\}$, of full measure.

We note that, under well-known restrictions on the function $f(S_t x)$, on the set M equation (1) is equivalent to

$$\frac{d\varphi(S_t x)}{dt} = f(S_t x) - \alpha. \tag{1'}$$

In the present note, conditions are given for the existence of a solution of equation (1), and some examples of the application of this equation to the study of dynamical systems are considered (see ([2–4])).

I. It is not difficult to show that, by virtue of the indecomposability of the measure m, the function $\varphi(x)$ is determined by equation (1) up to a constant term. Therefore, in the class of functions from $L^2_m(R)$ with zero mean value, the solution is unique. In particular, it follows from this that $\varphi(x)$ is real if $f(x)$ is real.

Theorem 1. *If there exists a function $\varphi_0(x) \in L^2_m(R)$ such that, for some $t_0 \neq 0$ and almost all $x \in R$*

$$\varphi_0(S_{t_0}x) = \varphi_0(x) + \int_0^{t_0} f^*(S_\tau x)\,d\tau, \quad \text{where } f^*(x) = f(x) - \alpha, \tag{2}$$

then there exists a solution of equation (1).

Proof. Obviously, one may assume that the mean value of $\varphi_0(x)$ is equal to 0. Then, in the space of functions from $L^2_m(R)$ orthogonal to 1, (2) will be equivalent to

$$U_{t_0}\varphi_0 = \varphi_0 + \int_0^{t_0} U_\tau f^*\,d\tau, \tag{3}$$

where U_t is the unitary operator corresponding to S_t. Let A be self-adjoint-

the infinitesimal operator of the group $\{U_t\}$, $E(\lambda)$ is its spectral function, and $g_1, \dots, g_k$ are generating vectors for A. Then

$$\varphi_0 = \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} b_k(\lambda)\,dE(\lambda)g_k, \qquad f^* = \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} a_k(\lambda)\,dE(\lambda)g_k,$$

where

$$\sum_{k=1}^{\infty} \int_{-\infty}^{\infty} |b_k(\lambda)|^2\,d\sigma_k(\lambda) < \infty, \qquad \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} |a_k(\lambda)|^2\,d\sigma_k(\lambda) < \infty,$$

where $\sigma_k(\lambda) = (E(\lambda)g_k, g_k)$. Further,

$$U_{t_0}\varphi_0 = \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} e^{i\lambda t_0} b_k(\lambda)\,dE(\lambda)g_k,$$

$$\int_0^{t_0} U_\tau f^*\,d\tau = \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i\lambda t_0} - 1}{i\lambda}\,a_k(\lambda)\,dE(\lambda)g_k.$$

Thus, (3) gives

$$\sum_{k=1}^{\infty} \int_{-\infty}^{\infty} (e^{i\lambda t_0} - 1) b_k(\lambda)\,dE(\lambda)g_k = \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i\lambda t_0} - 1}{i\lambda}\,a_k(\lambda)\,dE(\lambda)g_k,$$

whence, for any k,

$$\int_{\infty}^{\infty}(e^{i\lambda t_0}-1)b_k(\lambda)\,dE(\lambda)g_k=\int_{-\infty}^{\infty}\frac{e^{i\lambda t_0}-1}{i\lambda}\,a_k(\lambda)\,dE(\lambda)g_k.$$

This equality, in turn, means that almost everywhere with respect to the measure $d\sigma_k(\lambda)$,

$$(e^{i\lambda t_0}-1)b_k(\lambda)=\frac{e^{i\lambda t_0}-1}{i\lambda}\,a_k(\lambda).$$

Thus, for any k, for almost all λ (with respect to the measure $d\sigma_k(\lambda)$), either $b_k(\lambda)=a_k(\lambda)/i\lambda$, or $\lambda=2\pi n/t_0$, where n is an integer. Hence, for $\lambda\neq 0$,

$$\left|\frac{a_k(\lambda)}{\lambda}\right|^2\le|b_k(\lambda)|^2+\frac{|a_k(\lambda)|^2t_0^2}{4\pi^2n^2}\le|b_k(\lambda)|^2+t_0^2|a_k(\lambda)|^2,$$

and, consequently, by the continuity of $\sigma_k(\lambda)$ at zero,

$$\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\left|\frac{a_k(\lambda)}{\lambda}\right|^2\,d\sigma_k(\lambda)<\infty. \tag{4}$$

From (4) follows the existence of a function $\varphi(x)\in L_m^2(R)$ such that

$$\varphi=\frac{1}{i}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{a_k(\lambda)}{\lambda}\,dE(\lambda)g_k=\frac{1}{i}A^{-1}f^*.$$

As was shown in (3), it follows from this that, for any fixed t, for almost all $x\in R$,

$$\varphi(S_tx)=\varphi(x)+\int_0^t f^*(S_\tau x)\,d\tau. \tag{5}$$

With the aid of a well-known device (1), one can alter the function $\varphi(x)$ on a set of measure 0 so that equality (5) holds for any t at the points of some invariant set of full measure with respect to the system $\{S_t\}$. This means precisely that $\varphi(x)$ satisfies equation (1).

Theorem 2. In order that equation (1) have a solution in $L_m^2(R)$, it is necessary and sufficient that

$$\int_R\left(\frac{1}{t}\int_0^t f(S_\tau x)\,d\tau-\alpha\right)^2dm=O\left(\frac{1}{t^2}\right). \tag{6}$$

Proof. Suppose (6) is satisfied. Clearly, in $L^2_m(R)$ (6) is equivalent to

$$\left\|\int_0^t U_\tau f^*\,d\tau\right\| \le C, \tag{6'}$$

where C does not depend on t. If, for any $t_0 \ne 0$, we put

$$g(x) = \int_0^{t_0} f^*(S_\tau x)\,d\tau,$$

then

$$\left\|\sum_{k=0}^{n} U_{t_0}^k g\right\| \le C$$

for all n. From Browder' s results [5] it follows that in this case there exists $\varphi_0 \in L^2_m(R)$ such that

$$U_{t_0}\varphi_0 = \varphi_0 + g.$$

By the definition of g, this equality is equivalent to (2), and by Theorem 1 equation (1) has a solution.

Now let $\varphi \in L^2_m(R)$ be a solution of equation (1). In $L^2_m(R)$ this means that

$$U_t\varphi = \varphi + \int_0^t U_\tau f^*\,d\tau,$$

whence

$$\left\|\int_0^t U_\tau f^*\,d\tau\right\| \le \|U_t\varphi\| + \|\varphi\| = 2\|\varphi\|.$$

Thus $(6')$ is satisfied, and hence so is (6).

Remark. By virtue of the indecomposability of the measure m, it follows from the ergodic theorem that the left-hand side of (6) tends to zero as $t \to \infty$ for any function $f \in L^2_m(R)$.

II. Let now $\{\widetilde{S}_t\}$ be another dynamical system in R, obtained from the system $\{S_t\}$ by the change of time $d\tau = F(S_t x)\,dx$, where $F(x)$ is a real measurable function, bounded with respect to the measure m, such that $F(x) \ge \varepsilon > 0$. The measure $\widetilde{m}$, defined by the formula $d\widetilde{m} = F(x)\,dm$, is known to be invariant and indecomposable with respect to the system

$\{\widetilde{S}_t\}$. Assuming that both measures m and $\widetilde{m}$ are normalized, from Theorem 2 and the results of the remark [(2)] the following follows.

Theorem 3. If the function $F(x)$ satisfies condition (6), then the systems $\{S_t\}$ and $\{\widetilde{S}_t\}$, with measures m and $\widetilde{m}$, are isomorphic.

III. Consider the system of differential equations:

$$\frac{dx_k}{dt} = f_k(x_1, \dots, x_p), \qquad k = 1, 2, \dots, p, \tag{7}$$

where the f_k are real continuous functions with period 2π in each variable. Suppose that system (7) possesses a positive indecomposable integral invariant $M(x_1, \dots, x_p)$, having period 2π in each variable, and that uniqueness of solutions holds for it. In this case system (7) defines on the p-dimensional torus with cyclic coordinates, taken modulo 2π, a dynamical system with indecomposable measure $dm = M(x_1, \dots, x_p)\, dx_1, \dots, dx_p$. Put

$$\alpha_k = \frac{\int_0^{2\pi} \cdots \int_0^{2\pi} f_k M \, dx_1 \dots dx_p}{\int_0^{2\pi} \cdots \int_0^{2\pi} M \, dx_1 \dots dx_p}.$$

Then the following holds:

Theorem 4. If, for some k,

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \left(\frac{1}{t} \int_0^t f_k \, d\tau - \alpha_k \right)^2 M \, dx_1 \dots dx_p = O\left(\frac{1}{t^2}\right), \tag{8}$$

then α_k is an eigenfrequency of system (7).

Proof. By Theorem 2 there exists a real function $\varphi_k(x_1, \dots, x_p)$, of period 2π in each variable and of class L_m^2, such that along almost all trajectories of system (7)

$$\frac{d\varphi_k}{dt} = f_k - \alpha_k.$$

Put $\Phi_k(x_1, \dots, x_p) = e^{i(x_k - \varphi_k)}$. Then Φ_k will be an eigenfunction of system (7) with frequency α_k. Indeed,

$$\frac{d\Phi_k}{dt} = i\alpha_k \Phi_k.$$

Remark. If condition (8) is fulfilled for all $k = 1, 2, \dots, p$, then the discrete part of the spectrum of system (7) contains all frequencies of the form $n_1\alpha_1 + \dots + n_p\alpha_p$, where $n_1, \dots, n_p$ are integers.

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Note: Figure translations are in progress. See original paper for figures.