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Abstract

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THE DIRICHLET PROBLEM FOR AN ELLIPTIC SYSTEM OF TWO SECOND-ORDER DIFFERENTIAL EQUATIONS

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1. We consider the Dirichlet problem in a bounded simply connected two-dimensional domain D with boundary Γ for an elliptic system of two differential equations

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Eu_x + Fu_y + Ku = h(z), \tag{1}$$

where $u = (u_1, u_2)$ is the unknown vector; $h = (h_1, h_2)$ is a given real vector; A, B, C, E, F, K are square matrices of order 2, whose elements are real functions of $z = x + iy$. Suppose that the matrices A, B, C belong to the class C^2_α, the matrices E, F to the class C^1_α, the matrix K to the class C, and the vector h belongs to the class C.

For equations (1) the Dirichlet problem is posed as follows: in the domain D with boundary Γ, one is required to find a regular solution of equation (1), belonging to the class $C'_\alpha(\bar{D})$ and satisfying the boundary condition

$$u|_\Gamma = g(z), \tag{2}$$

where $g(z)$ is a given vector on Γ of class $C^1_\alpha(\Gamma)$. The problem (1)–(2) with $h \equiv 0$, $g \equiv 0$ will be called the homogeneous problem.

It is known ([1]) that, if system (1) is strongly elliptic, then the Dirichlet problem for the adjoint system forms a Fredholm pair of boundary-value problems. For general elliptic systems this assertion, generally speaking, does not hold. As examples show ([2]), the homogeneous problem (1)–(2) for elliptic systems may admit an infinite number of linearly independent solutions.

Consider the characteristic matrix of system (1)

$$A + 2B\lambda + C\lambda^2 = \|a_{ij}(\lambda)\|. \tag{3}$$

In what follows, without loss of generality, we shall assume that the matrix C is the identity matrix.

It is proved in the article ([3]) that the Dirichlet problem for system (1) is Fredholm also in the case when the imaginary parts of the roots of the equation with respect to λ

$$a_{11}(\lambda) + a_{22}(\lambda) + i(a_{21}(\lambda) - a_{12}(\lambda)) = 0 \tag{4}$$

have different signs. From the ellipticity of system (1) it follows that the roots of equation (3) cannot lie on the x-axis, and therefore the fulfillment of this condition is sufficient to verify at only one point of the domain D.

We shall agree to say that problem (1)–(2) is **normally solvable** if the homogeneous problem (1)–(2) has a finite number of linearly independent solutions and if, for the solvability of the inhomogeneous problem, h and g must satisfy a finite number of orthogonality conditions.

The Dirichlet problem and more general boundary-value problems for elliptic systems are considered in ([4],[7]). The theorems stated there on the necessary and sufficient condition for normal solvability of the posed problems are, unfortunately, erroneous (see ([4]), pp. 75, 107, 149, 185, 199; ([7]), pp. 60, 72, 81, 86). This is easily verified on the example of the equation

$$\frac{\partial}{\partial z}\left[\left(\frac{1-z}{4}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial \bar z}\right)\beta\right] = 0, \tag{5}$$

where $\beta = 16(16 - |1-z|^2)^{-1}$, $\partial/\partial\bar z = \frac{1}{2}(\partial/\partial x + i\,\partial/\partial y)$, $\partial/\partial z = \frac{1}{2}(\partial/\partial x - i\,\partial/\partial y)$, $u = u_1 + iu_2$, $\bar u = u_1 - iu_2$.

Equation (5) is the complex form of writing a system of two equations of elliptic type in the disk $|z| < 1$. The Dirichlet problem for equation (5) in the disk $|z| < 1$ does not satisfy the necessary and sufficient condition for normal solvability formulated in ([4]), p. 185, but nevertheless it always has a solution, and this solution is unique. This follows directly from the general representation of the solutions of system (5)

$$u = \Phi'(z)\bar z - \frac{1 - z\bar z}{4}\overline{\Phi(z)} + \Psi(z), \qquad \Phi(0) = 0, \tag{6}$$

where Φ and Ψ are holomorphic functions in the unit disk (here and below the bar denotes passage to complex conjugate quantities).

If the first derivatives of the unknown functions enter into one of the boundary conditions of system (1), while they do not enter into the second boundary condition, then, by the results of ([4]), it follows that for such a problem normal solvability does not occur; this also requires clarification. It should also be noted that under certain conditions imposed on the coefficients of system (1), in ([4],[7]) the Dirichlet problem is reduced to a singular integral equation of normal type, but complete equivalence between them is not established.

2. Let λ_1 and λ_2 be the roots of the characteristic equation

$$|A + 2B\lambda + C\lambda^2| = \begin{vmatrix} a_{11}(\lambda) & a_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) \end{vmatrix} = 0 \tag{7}$$

with positive imaginary parts. It is not hard to prove that throughout the closed domain D one of the following inequalities holds:

$$\left|a_{11}(\lambda_1) + a_{22}(\lambda_1) + (\lambda_1 - \bar{\lambda}_1)(\bar{\lambda}_2 - \lambda_1)\right| >$$

$$> \left|a_{11}(\lambda_1) + a_{22}(\lambda_1) + (\lambda_1 - \bar{\lambda}_1)(\lambda_2 - \lambda_1)\right|; \tag{8}$$

$$\left|a_{11}(\lambda_1) + a_{22}(\lambda_1) + (\lambda_1 - \bar{\lambda}_1)(\bar{\lambda}_2 - \lambda_1)\right| <$$

$$< \left|a_{11}(\lambda_1) + a_{22}(\lambda_1) + (\lambda_1 - \bar{\lambda}_1)(\lambda_2 - \lambda_1)\right|. \tag{9}$$

Without loss of generality we may suppose that $g \equiv 0$.

Under the condition that

$$a_{11}(\lambda_1) + a_{22}(\lambda_1) + (\lambda_1 - \bar{\lambda}_1)(\bar{\lambda}_2 - \lambda_1) \neq 0, \qquad z \in \Gamma, \tag{10}$$

the following two theorems hold.

Theorem 1. The number k of linearly independent solutions of the homogeneous Dirichlet problem for system (1) and the number k' of linearly independent solutions of the homogeneous Dirichlet problem for the adjoint system* of equations are finite, and

$$k - k' = \begin{cases} 0, & \text{in the case of inequality (8)}, \\ \dfrac{1}{\pi} \Delta_\Gamma \left(|a_{11}(\lambda_1)|^2 + a_{21}(\lambda_1)\overline{a_{12}(\lambda_1)}\right), & \text{in the case of inequality (9)}; \end{cases} \tag{11}$$

here $\Delta_\Gamma \left(|a_{11}(\lambda_1)|^2 + a_{21}(\lambda_1)\overline{a_{12}(\lambda_1)}\right)$ denotes the increment of the function

$$\arg \left(|a_{11}(\lambda_1)|^2 + a_{21}(\lambda_1)\overline{a_{12}(\lambda_1)}\right),$$

when z describes once the boundary Γ of the domain D in the positive direction.

Theorem 2. In order that the nonhomogeneous Dirichlet problem have a solution, it is necessary and sufficient that the conditions

$$\iint_D (h_1 V_{i1} + h_2 V_{i2})\, dS = 0, \qquad i = 1, \dots, k', \tag{12}$$

be satisfied, where $(V_{11}, V_{12}), \dots, (V_{k'1}, V_{k'2})$ is a complete system of linearly independent solutions of the homogeneous Dirichlet problem for the adjoint system.

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* By "the system adjoint to system (1)" we mean in the sense of Lagrange.

If condition (10) is violated at a finite number of boundary points $z_1, \dots, z_n$ and, in a neighborhood of each of them,

$$a_{11}(\lambda_1) + \overline{a_{22}(\lambda_1)} + (\lambda_1 - \overline{\lambda_1})(\overline{\lambda_2} - \lambda_1) = (z - z_k)^{n_k} f_k(z) \quad (k = 1, 2, \dots, n), \tag{13}$$

where n_k are integers, $f_k(z_k) \neq 0$, and $f_k(z)$ satisfies the Hölder condition on Γ, then the Dirichlet problem for system (1) still remains normally solvable. On the other hand, it is not difficult to construct an example of an elliptic system for which condition (10) is violated on a set of points of Γ of positive, arbitrarily small measure, in such a way that the Dirichlet problem for this system ceases to be normally solvable.

It is of interest to clarify whether the Dirichlet problem for system (1) remains normally solvable when condition (10) is violated on a countable set of points of Γ. In analyzing condition (10) it turned out that it coincides with the condition obtained in the work ([8]).

In the work ([7]) a method is given for computing the index of the first boundary-value problem (i.e., for $k - k'$), and it is asserted that this index for elliptic systems depends essentially not only on the values of the coefficients of the system on the boundary of the domain, but also on the values of the coefficients inside the domain. From our Theorem 1 it follows that, when the condition of Ya. B. Lopatinskii ([8]) is fulfilled, in the case of elliptic systems of two second-order differential equations the index depends only on the values of the coefficients on the boundary of the domain.

3. When condition (10) is fulfilled, the Dirichlet problem is reduced to an equivalent singular integral equation of normal type.

Suppose that there exists a point $z = z_0$ in the closed domain $\overline{D}$, where

$$\operatorname{Im}\left(a_{11}(\lambda_1)\overline{a_{21}(\lambda_1)}\right)\Big|_{z=z_0} = 0. \tag{14}$$

Three cases are possible:

a) $a_{11}(\lambda_1)\big|_{z=z_0} = 0, \quad a_{12}(\lambda_1)\big|_{z=z_0} = 0;$

b) $a_{11}(\lambda_1)\big|_{z=z_0} = 0, \quad a_{21}(\lambda_1)\big|_{z=z_0} = 0;$

c) $a_{11}(\lambda_1)\big|_{z=z_0} \neq 0.$

In case c), we first multiply both sides of system (1) by the matrix

$$\begin{Vmatrix} -k_0 & 1 \\ 1 & 0 \end{Vmatrix}, \qquad k_0 = \frac{a_{21}(\lambda_1)}{a_{11}(\lambda_1)}\bigg|_{z=z_0}.$$

Then, by means of the corresponding substitutions:

a) $v_1 = u_1,\ v_2 = \alpha u_2;$
b) $v_1 = \alpha u_1,\ v_2 = u_2;$
c) $v_1 = \alpha u_1,\ v_2 = -k_0 u_1 + u_2$

we obtain a system for which, for small α, at the point $z = z_0$ equation (4) will have roots with imaginary parts of different signs. Consequently (see ([3])), under condition (14) the Dirichlet problem for system (1) is Fredholm. If condition (14) is satisfied, then inequality (8) always holds.

Suppose that

$$\operatorname{Im}(a_{11}(\lambda_1)\overline{a_{21}(\lambda_1)}) \neq 0, \qquad z \in \overline{D}, \tag{15}$$

and that inequality (9) holds. Multiplying the first equation of system (1) by $a_{21}(\lambda_1)$, and the second by $-a_{11}(\lambda_1)$, and adding them, we obtain the equation

$$\frac{\partial}{\partial \xi_2}\left(q_1 \frac{\partial v}{\partial \xi_1} + \frac{\partial \overline{v}}{\partial \xi_1}\right) = F_1(h, v, v_x, v_y), \tag{16}$$

where

$$\begin{aligned} v = v_1 + iv_2 = [a_{11}(\lambda_1)(k_1 - \overline{k_1}) - (\lambda_1 - \overline{\lambda_1})(\lambda_2 - \lambda_1)\overline{k_1}]u_1 \\ + [a_{12}(\lambda_1)(k_1 - \overline{k_1}) + (\lambda_1 - \overline{\lambda_1})(\lambda_2 - \lambda_1)]u_2, \end{aligned} \tag{17}$$

$$k_1 = \frac{a_{21}(\lambda_1)}{a_{11}(\lambda_1)}, \qquad \frac{\partial}{\partial \xi_1} = \frac{\partial}{\partial z} + \alpha_1 \frac{\partial}{\partial \overline{z}}, \qquad \frac{\partial}{\partial \xi_2} = \frac{\partial}{\partial z} + \alpha_2 \frac{\partial}{\partial \overline{z}}, \qquad \alpha_1 = \frac{\overline{\lambda_2} + i}{\overline{\lambda_2} - i},$$

$$\alpha_2 = \frac{\lambda_1 - i}{\lambda_1 + i}, \qquad q_1 = \frac{(\overline{\lambda_2} - \lambda_1)(\lambda_2 + i)}{(\lambda_2 - \overline{\lambda_1})(\overline{\lambda_2} - i)} \, \frac{a_{11}(\lambda_1) + a_{22}(\lambda_1) + (\lambda_1 - \overline{\lambda_1})(\overline{\lambda_2} - \lambda_1)}{a_{11}(\lambda_1) + \overline{k_1} a_{12}(\lambda_1)}.$$

F_1 is a linear function of its arguments; moreover, there is complete equivalence between equations (1) and (16).

As a result of the change of the independent variable $\tau = \xi + i\eta = w_1(z)$, where $w_1(z)$ is the principal homeomorphism of the equation $\partial w/\partial \xi_1 = 0$ (see (5), p. 96), equation (16) takes the form

$$\frac{\partial}{\partial \xi_2}\left(q_2 \frac{\partial v}{\partial \tau} + \frac{\partial \bar{v}}{\partial \tau}\right) = F_2(h, v, v_\xi, v_\eta), \qquad \tau \in D_1, \tag{18}$$

where D_1 is the image of the domain D under the mapping $\tau = w_1(z)$,

$$\frac{\partial}{\partial \xi_3} = \frac{\partial}{\partial \tau} + \alpha_3 \frac{\partial}{\partial \bar{\tau}}, \qquad \alpha_3 = \frac{\alpha_2 - \bar{\alpha}_1}{1 - \alpha_1 \alpha_2}\left(\frac{\partial \bar{w}_1}{\partial z}\right)\left(\frac{\partial w_1}{\partial z}\right)^{-1}, \qquad q_2 = q_1 \frac{\partial \bar{w}_1}{\partial z}\left(\frac{\partial w_1}{\partial z}\right)^{-1},$$

and F_2 is a linear function of its arguments.

Let us first suppose that F_2 does not depend on v, v_ξ, v_η. Then the general solution of equation (18) is written in the form:

$$v = \frac{1}{\pi} \iint_D \frac{\Phi(w_2(t))}{t - \tau} \frac{\partial}{\partial \bar{t}} q_3(t)\, dS_t - \frac{1}{\pi} \iint_D \frac{\Phi'(w_2(t))\, dS_t}{(t - \tau)(1 - |q_2(t)|^2)} +$$

$$+ q_3(\tau)\Phi(w_2(\tau)) + \Psi(\tau) + u_0(F_2), \tag{19}$$

where $u_0(F_2)$ is a particular solution of equation (18), $\Psi(\tau)$ is an arbitrary analytic function in the domain D_1; $w_2(\tau)$ is the principal homeomorphism of the equation $\partial w/\partial \xi_3 = 0$, $\Phi(\zeta)$ is an arbitrary analytic function in the domain D_2, with $\Phi(\zeta_0) = 0$; D_2 is the image of D_1 under the mapping $\zeta = w_2(\tau)$, and ζ_0 is a fixed point in the domain D_2; $q_3(\tau) = -q_2[(1 - |q_2|^2)\partial w/\partial \bar{\tau}]^{-1}$. (19) establishes a one-to-one correspondence between v and the pair of analytic functions Φ and Ψ.

Since the solution is sought in the class $C^1_\alpha(\bar{D}_1)$, it follows from (19) that $\Phi \in C^1_\alpha(\bar{D}_2)$, $\Psi \in C_\alpha(\bar{D}_1)$.

If condition (10) is satisfied, then, using the integral representation of analytic functions (see (6))

$$\Psi = \frac{1}{\pi i} \int_{\Gamma_1} \frac{\mu_1(t)\, dt}{t - \tau} + ic_1, \qquad \Phi(w_2(\tau)) = \frac{1}{\pi i} \int_{\Gamma_1} \frac{\mu_2(t)\, dw_2(t)}{w_2(t) - w_2(\tau)} + ic_2,$$

where $\mu_1(t)$ and $\mu_2(t)$ are real functions, and c_1 and c_2 are real constants, the Dirichlet problem for equation (18) is easily reduced to a system of singular integral equations of normal type (Γ_1 is the boundary of the domain D_1).

Let now F_2 depend on v, v_ξ, v_η. Using the results set forth above, the Dirichlet problem in this case can be reduced to the solution of a Fredholm equation and Theorems 1 and 2 can be proved. If, however, inequalities (15) and (8) hold, then the Dirichlet problem is reduced, in an entirely analogous way, again to an equivalent integral equation of normal type; only in this case, instead of (17), one must take the substitution

$$v = v_1 + iv_2 = [\overline{a_{11}(\lambda_1)}(k_1 - \bar{k}_1) - \bar{k}_1(\lambda_1 - \bar{\lambda}_1)(\lambda_2 - \bar{\lambda}_1)]u_1 +$$

$$+ [\overline{a_{12}(\lambda_1)}(k_1 - \bar{k}_1) + (\lambda_1 - \bar{\lambda}_1)(\lambda_2 - \bar{\lambda}_1)]u_2.$$

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Note: Figure translations are in progress. See original paper for figures.