ON THE APPROXIMATION OF FUNCTIONS PRESCRIBED OUTSIDE AN INTERVAL BY ENTIRE FUNCTIONS OF FINITE DEGREE

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Abstract

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ON THE APPROXIMATION OF FUNCTIONS PRESCRIBED OUTSIDE AN INTERVAL BY ENTIRE FUNCTIONS OF FINITE DEGREE

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A. F. Timan (5), and also (6), Ch. V, § 5.2, proved a theorem strengthening the classical theorem of Jackson. V. K. Dzyadyk (4) and G. Freud (8) generalized A. F. Timan' s theorem to classes of functions with a prescribed second modulus of smoothness of the r-th derivative. Recently Yu. A. Brudnyi (3) generalized this theorem to classes of bounded functions with derivative differential-difference properties. In the works of Yu. A. Brudnyi (1) and R. M. Trigub (7), theorems similar to the theorems of A. F. Timan and V. K. Dzyadyk–G. Freud were obtained for functions prescribed outside an interval. In the present work these results are generalized to the case of bounded functions with arbitrary differential-difference properties.

The functions considered here are assumed not to coincide with a constant on at least one connected component of their domain of definition.

Theorem 1. *Let $f(x)$ be a function prescribed and bounded on $(-\infty;-1]\cup[1;+\infty)$. For any $\sigma>0$ there exists an entire function $G_\sigma(x)$ of degree $\leqslant\sigma$ such that*

$$|f(x)-G_\sigma(x)|\leqslant M\omega_r\left(f;\frac{\sqrt{x^2-1}}{|x|\sigma}+\frac{1}{\sigma^2}\right),$$

where M does not depend on σ and x, and $\omega_r(f;h)$ is the r-th modulus of smoothness of $f(x)$ (see (6)).

The proof is carried out by a method analogous to that of (3).

Remark. The constant M depends on the function f. The estimate

$$M\leqslant M'\frac{\sup_{|x|\geqslant 1}f(x)-\inf_{|x|\geqslant 1}f(x)}{\omega_r(f;2)},$$

holds, where M' is a constant depending only on r.

Corollary. *Let $f(x)$ be a function prescribed on $(-\infty;-1]\cup[1;+\infty)$ and having in its domain of definition a k-th bounded derivative $f^{(k)}(x)$. For any $\sigma>0$ there exists an entire function $G_\sigma(x)$ of degree $\leqslant\sigma$ such that*

$$|f(x)-G_\sigma(x)|\leqslant M\left(\frac{\sqrt{x^2-1}}{|x|\sigma}+\frac{1}{\sigma^2}\right)^k\omega_r\left(f^{(k)};\frac{\sqrt{x^2-1}}{|x|\sigma}+\frac{1}{\sigma^2}\right),$$

where M does not depend on σ and x.

For a function prescribed and bounded outside an arbitrary interval $(a;b)$, the corresponding estimate has the form

$$|f(x)-G_\sigma(x)|\leqslant M\left(\frac{\lambda(x)}{\sigma}+\frac{1}{\sigma^2}\right)^k\omega_r\left(f^{(k)};\frac{\lambda(x)}{\sigma}+\frac{1}{\sigma^2}\right),$$

where

$$\lambda(x)=\sqrt{(x-a)(x-b)}\Big/\left|x-\frac{a+b}{2}\right|.$$

Theorem 2. Let $f(x)$ be a function defined and bounded on

$$D=(-\infty;b]\cup[a_1,b_1]\cup\cdots\cup[a_m,b_m]\cup[a;+\infty).$$

$$(b<a_1<b_1<a_2<\cdots<b_m<a).$$

For every $\sigma>0$ there exists an entire function $G_\sigma(x)$ of degree $\leqslant\sigma$ such that

$$|f(x)-G_\sigma(x)|\leqslant M\omega_r\left(f;\frac{\lambda(x)}{\sigma}+\frac{1}{\sigma^2}\right),$$

where M is a constant independent of σ and x,

$$\lambda(x)=\begin{cases}\dfrac{\sqrt{(x-a)(x-b)}}{|x|+1}, & \text{for } x<b \text{ and } x>a;\\ \sqrt{(x-a_i)(b_i-x)}, & \text{for } x\in[a_i;b_i].\end{cases}$$

Theorem 3. Let $f(x)$ be a function defined and bounded on

$$D=\bigcup_{j=-\infty}^{\infty}[a_i,b_i]$$

$$(a_i < b_i;\ b_{i+1} - a_i \geqslant d > 0;\ i = 0; \pm 1; \pm 2; \ldots).$$

For every $\sigma > 0$ there exists an entire function $G_\sigma(x)$ of degree $\leqslant \sigma$ such that

$$|f(x) - G_\sigma(x)| \leqslant M\omega_r\left(f; \frac{\lambda(x)}{\sigma} + \frac{1}{\sigma^2}\right),$$

where M is a constant independent of σ and x,

$$\lambda(x) = \frac{\sqrt{(x - a_i)(b_i - x)}}{b_i - a_i} \quad \text{for } x \in [a_i; b_i].$$

From this theorem, Theorem 1″ of paper ([2]) follows immediately. I consider it my pleasant duty to express gratitude to Prof. A. F. Timan and Yu. A. Brudnyi for their attention to the present work.

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Note: Figure translations are in progress. See original paper for figures.