THEORY OF ELASTICITY

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Abstract

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ON THE ANALYTIC STRUCTURE OF THE SOLUTION OF THE MEMBRANE EQUATION

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The so-called membrane equation of a circular thin plate is investigated ([1])

$$32p^2(x)[xp(x)]'' - 1 = 0, \tag{1}$$

$0 \le x \le 1$, where $p(x)$ is proportional to the radial force and opposite to it in sign.

For $x = 1$ we have three possible boundary conditions:

$$-p(1) = T > 0; \tag{a}$$

$$p(1) = 0; \tag{b}$$

$$2p'(1) + (1 - \sigma)p(1) = 0, \tag{c}$$

corresponding respectively to tension on the contour, absence of shearing forces on the contour, and absence of radial contour displacements.

Following Hencky ([2]), we seek a formal solution of the equation in the form of a power series in x

$$p(x) = \frac{1}{2}\sum_{n=0}^{\infty} a_n x^n. \tag{2}$$

The coefficients a_n are determined through a_0:

$$a_1 = \frac{1}{a_0^2}, \qquad a_2 = -\frac{2}{3a_0^5}, \dots . \tag{3}$$

We shall establish the possibility of such a choice of a_0 that the series (2) converges on $[0, 1]$ and its sum satisfies conditions (a) or (c).

§ 1. **Investigation of problems** (1)–(a) and (1)–(c). In ([3]) it was obtained that the possible solutions of problems (1)–(a), (1)–(b), and (1)–(c) are nonpositive. In ([4]) the uniqueness of the solution of the indicated problems and the existence of solutions of problems (1)–(a) and (1)–(c) in the class of twice continuously differentiable functions were proved.

Let us pass, for problems (1)–(a) and (1)–(c), to integral equations

$$p(x) = \frac{1}{x}\left(\int_0^x \int_0^{x_0} \frac{d\tilde{x}\, dx_0}{32p^2(\tilde{x})} - \int_0^1 \int_0^{x_0} \frac{d\tilde{x}\, dx_0}{32p^2(\tilde{x})}\right) - T, \tag{4a}$$

$$p(x) = \frac{1}{x}\int_0^x \int_0^{x_0} \frac{d\tilde{x}\, dx_0}{32p^2(\tilde{x})} + \frac{1+\sigma}{1-\sigma}\int_0^1 \int_0^{x_0} \frac{d\tilde{x}\, dx_0}{32p^2(\tilde{x})} - \frac{2}{1-\sigma}\int_0^1 \frac{dx}{32p^2(x)}. \tag{4c}$$

We shall consider equation (4a) (for (4c) we obtain analogous results). Following (4), we make the change of variables $\tilde{p}(x) = p(x) + T$,

$$\tilde{p}(x) = \frac{1}{x}\int_0^x \int_0^{x_0} \frac{d\tilde{x}\, dx_0}{32[T-\tilde{p}(\tilde{x})]^2} - \int_0^1 \int_0^{x_0} \frac{d\tilde{x}\, dx_0}{32[T-\tilde{p}(\tilde{x})]^2} = \Phi(\tilde{p}). \tag{5}$$

In paper (4) it was proved that there exists a continuous nonpositive $\tilde{p}_0(x)$ satisfying equation (5).

Consider the space l of power series of the form $v = \sum_{n=0}^{\infty} c_n x^n$, convergent on the segment $[0,1]$, and introduce in l the norm $\max_{0\le x\le 1} |v(x)|$. The resulting space will be a Banach space. Consider the series satisfying the conditions:

$$1)\ c_0 \le 0; \qquad 2)\ c_0 c_k \le 0\ (k = 1, 2, ...); \qquad 3)\ |c_0| \ge \sum_{k=1}^{\infty} c_k.$$

These series form a cone K in the space l.

Let S_R be the sphere with center at the origin in the space l, and let $K_R = S_R \cap K$. K_R is convex as the intersection of two convex sets. One can choose $R = R_0$ so that the operator Φ maps the set K_{R_0} into itself. The operator Φ is completely continuous in the space l, and by the Schauder principle we obtain that equation (5) has a solution from the set K_{R_0}

$$\tilde{p}_0(x) = \sum_{n=0}^{\infty} c_n x^n. \tag{6}$$

Thus, the solution of problem (1)–(a) (as well as of problem (1)–(c)) is always representable in the form of a power series convergent on the segment $[0,1]$

and satisfying the boundary condition at $x = 1$. Substituting series (6) into equation (1) and choosing $a_0 = 4c_0$, we find that the coefficients of series (2) and (6) coincide.

§ 2. **Existence of a solution of problem (1)–(b).** Let $xp = U$. Equation (1) is transformed to the form

$$U''(x) = \frac{x^2}{32U^2(x)} \tag{7}$$

with boundary conditions $U(0) = U(1) = 0$. Problem (1)–(b) reduces to the integral equation

$$U(x) = \int_0^x dt \int_0^t \frac{\xi^2\, d\xi}{32U^2(\xi)} - x \int_0^1 dt \int_0^t \frac{\xi^2\, d\xi}{32U^2(\xi)}. \tag{8}$$

Take

$$U_1 = -\frac{\sqrt[3]{9}}{4}\, x(1-x)^{2/3}$$

and consider the process of successive approximations

$$U''_{n+1}(x) = \frac{x^2}{32U_n^2(x)}, \qquad U_{n+1}(0) = U_{n+1}(1) = 0. \tag{9}$$

It can be proved, analogously to what was done in paper (5), that

$$U_1(x) \le U_2(x) \le \cdots \le 0$$

and U_n converges uniformly to some function $U_0(x)$ on the segment $[0,1]$, with $U_0(x) < 0$ for $x \in (0,1)$.

§ 3. Investigation of problem (1)–(b). Let us take an arbitrary x_1 $(0 < x_1 < 1)$. Since $U_0(x_1) < 0$, on the interval $[0, x_1]$ $U_0(x)$, as the solution of problem (1)–(a), can be represented in the form of an absolutely convergent power series of the form (6).

We now investigate the solution $U_0(x)$ in a neighborhood of the point $x = 1$. From equality (8) one can obtain that, for all n,

$$U_n(x) = -\frac{\sqrt[3]{9}}{4}\, x(1-x)^{2/3} y_n(x),$$

where $y_n(x)$ is continuous on $[0,1]$. It can be shown that $y_n(x)$ converges uniformly on $[0,1]$ to a certain function $y_0(x)$. Obviously,

$$U_0(x) = -\frac{\sqrt[3]{9}}{4}\, x(1-x)^{2/3} y_0(x), \tag{10}$$

On further investigation of the integral equality (8), we obtain

$$\underset{x\to 1}{y_0(x) \to 1}, \qquad \underset{x\to 1}{y_0'(x) \to -{}^2/_3}, \qquad \underset{x\to 1}{y_0''(x) \to {}^5/_{11}}.$$

Finally, for $p(x)$ one can write formulas of asymptotic character as x tends to unity:

$$p(x) = \frac{\sqrt[3]{9}}{4}(1-x)^{2/3}\left[-1 - {}^2/_3(1-x) + {}^5/_{22}(1-x)^2 + O[(1-x)^3]\right],$$

(11)

$$p'(x) = \frac{\sqrt[3]{9}}{4}(1-x)^{-1/3}\left[{}^2/_3 + {}^{10}/_9(1-x) - {}^{20}/_{33}(1-x)^2 + O[(1-x)^3]\right].$$

The results obtained justify the form proposed by Bromberg[1] for finding solutions of problem (1)–(b).

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Note: Figure translations are in progress. See original paper for figures.