DIFFERENTIAL PROPERTIES OF THE TRAJECTORIES OF DIFFUSION PROCESSES DEPENDING ON PARAMETERS, AND THE CAUCHY PROBLEM FOR DEGENERATE PARABOLIC EQUATIONS

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Abstract

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DIFFERENTIAL PROPERTIES OF THE TRAJECTORIES OF DIFFUSION PROCESSES DEPENDING ON PARAMETERS, AND THE CAUCHY PROBLEM FOR DEGENERATE PARABOLIC EQUATIONS

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In this note we consider diffusion processes $x_t(a, \omega)$ depending on a numerical parameter a and connected with equations of the form

$$\frac{\partial v}{\partial t} = \frac{1}{2} b^2(t, x) \frac{\partial^2 v}{\partial x^2} + a(t, x) \frac{\partial v}{\partial x}. \tag{1}$$

For the trajectories of such processes, estimates are found—optimal in a certain sense—of their differential properties with respect to a, both in mean square and with probability 1.

These results contain, as special cases, most of the results of (1,2) and lead to a priori estimates of the differential properties of the classical solution of the Cauchy problem for equation (1). The a priori estimates thus obtained depend only on the differential properties of the functions $b(t, x)$, $a(t, x)$, and $v(0, x)$, and do not depend on the "degeneracy" of $b(t, x)$, i.e., the Cauchy problem for equation (1) is considered in the case where $b(t, x)$ may vanish in an arbitrary manner.

Let $\xi_t(\omega)$, $t \in [0, T]$, denote the Wiener process (3, p. 187), defined on the probability space $(\Omega, \mathfrak{M}, \mathbf{P})$, and let $\mathfrak{F}_t$ denote the σ-algebra generated by the sets

$$\{\omega : \xi_{t_1}(\omega) < x_1, \dots, \xi_{t_N}(\omega) < x_N;\ t_1, \dots, t_N \in [0, t]\},$$

where $x_1, \dots, x_N$ are arbitrary numbers on the line $R = (-\infty, \infty)$; Ω is the space of elementary events ω; $\mathfrak{M}$ is the σ-algebra of subsets of the set Ω on which the probability measure $\mathbf{P}$ is defined.

We shall call a process $x_t(\omega)$ admissible if $x_s(\omega)$ is $\mathfrak{F}_{t_i}$-measurable for every $s \leq t_i$; we shall call a process $x_t(\omega)$, $t \in [0, T]$, diffusive if it is admissible, continuous in $t \in [0,T]$ for almost all $\omega \in \Omega$, and, for some functions $a(t,x)$, $b(t,x)$, $t \in [0,T]$, $x \in R$, with probability 1 the equality

$$x_t(\omega) = x_s(\omega) + \int_s^t a(u, x_u(\omega))\, du + \int_s^t b(u, x_u(\omega))\, d\xi_u(\omega) \tag{2}$$

is satisfied.

For the theory of such equations see, for example, (^{4}) (or (^{5}), Ch. VI, § 3). By $\partial x(a,\omega)/\partial a$ we shall denote a random variable $\eta(a,\omega)$ such that

$$\lim_{h\to 0} \mathbf{M} \left[\frac{x(a+h,\omega) - x(a,\omega)}{h} - \eta(a,\omega) \right]^2 = 0.$$

Here

$$\mathbf{M}\eta(\omega) = \int_\Omega \eta(\omega)\, dP$$

is the mathematical expectation of the random variable $\eta(\omega)$.

Let $x_0(\omega) \equiv x$ in equation (2). Under certain conditions on $a(t,x)$, $b(t,x)$, $\psi(x)$ (see (6)), the function $v(t,x) = \mathbf{M}\psi(x_t(\omega))$ is a solution of the Cauchy problem for equation (1) with initial condition $v(0,x) = \psi(x)$. Finally, let $x_t(\alpha,\omega)$, for each $\alpha \in D \subseteq \mathbf{R}$, be a diffusion process and satisfy almost surely (a.s.) the equation

$$x_t(\alpha,\omega) = x_0(\alpha,\omega) + \int_0^t a(u, x_u(\alpha,\omega), \alpha)\, du +$$

$$+ \int_0^t b(u, x_u(\alpha,\omega), \alpha)\, d\xi_u(\omega). \tag{3}$$

Theorem 1. If for any $x, y \in \mathbf{R}$, $\alpha, \beta \in D$, $t \in [0,T]$ there exist constants $\gamma \in (0,1)$, $L_{1r}, L_{2r} \in (0,\infty)$ such that

$$\sum_{k=0}^{m} \left[\left| \frac{\partial^m a(t,x,\alpha)}{\partial x^k \partial \alpha^{m-k}} - \frac{\partial^m a(t,y,\beta)}{\partial y^k \partial \beta^{m-k}} \right| + \left| \frac{\partial^m b(t,x,\alpha)}{\partial x^k \partial \alpha^{m-k}} - \frac{\partial^m b(t,y,\beta)}{\partial y^k \partial \beta^{m-k}} \right| \right] \le$$

$$\le L_{10} (|x-y|^2 + |\alpha - \beta|^2)^{\gamma/2}, \tag{4}$$

$$\sum_{l=0}^{m} \sum_{k=0}^{l} \left[\left| \frac{\partial^l a(t,x,\alpha)}{\partial x^k \partial \alpha^{l-k}} \right| + \left| \frac{\partial^l b(t,x,\alpha)}{\partial x^k \partial \alpha^{l-k}} \right| \right] \le L_{20}, \tag{5}$$

$$\mathbf{M}\left[\frac{\partial^m x_0(\alpha,\omega)}{\partial\alpha^m}-\frac{\partial^m x_0(\beta,\omega)}{\partial\beta^m}\right]^{2r}\le L_{1r}|\alpha-\beta|^{2r\gamma};\tag{6}$$

$$\sum_{l=0}^{m}\mathbf{M}\left[\frac{\partial^l x_0(\alpha,\omega)}{\partial\alpha^l}\right]^{2r}\le L_{2r},\tag{7}$$

$$r=0,1,2,\dots,$$

then for $x_t(\alpha,\omega)$, $t\in[0,T]$, inequalities (6) and (7) hold with constants L'_{1r} and L'_{2r}, $r=1,2,\dots$, depending only on m,γ,L_{1r},L_{2r},T, $r=0,1,2,\dots$.

Theorem 2. If for each $r<\infty$ and bounded domain $D'\subseteq D$ there exist: a constant $L_r(D')$ such that for all $t\in[0,T]$, $x,y\in[-r,r]$, $\alpha,\beta\in D'$ inequality (4) holds with $L_{1r}=L_r(D')$, and a nonnegative random variable $L(D',\omega)<\infty$ a.s. such that for all $\alpha,\beta\in D'$

$$\left|\frac{\partial^m x_0(\alpha,\omega)}{\partial\alpha^m}-\frac{\partial^m x_0(\beta,\omega)}{\partial\beta^m}\right|\le L(D',\omega)|\alpha-\beta|^{\gamma},\tag{8}$$

then for $x_t(\alpha,\omega)$ inequality (8) holds with any $\gamma_1<\gamma$ and with a random variable $L'(D',\omega)<\infty$, depending only on $L(D',\omega)$, $L_r(D')$, T, m, γ, γ_1.

Theorem 3. If $a(t,x)$, $b(t,x)$, $\psi(x)$ are such that

$$\left|\frac{\partial^m a(t,x)}{\partial x^m}-\frac{\partial^m a(t,y)}{\partial y^m}\right|+\left|\frac{\partial^m b(t,x)}{\partial x^m}-\frac{\partial^m b(t,y)}{\partial y^m}\right|+$$

$$+\left|\frac{\partial^m\psi(x)}{\partial x^m}-\frac{\partial\psi(y)}{\partial y^m}\right|\le K_1|x-y|^{\gamma};\tag{9}$$

$$\sum_{l=0}^{m}\left(\left|\frac{\partial^l a(t,x)}{\partial x^l}\right|+\left|\frac{\partial^l b(t,x)}{\partial x^l}\right|+\left|\frac{\partial^l\psi(x)}{\partial x^l}\right|\right)\le K_2,\qquad m\ge 2,\ \gamma>0,\tag{10}$$

then for the classical solution $v(t,x)$ of equation (1) with initial condition $v(0,x)=\psi(x)$ the inequalities hold:

$$\left|\frac{\partial^m v(t,x)}{\partial x^m}-\frac{\partial^m v(t,y)}{\partial y^m}\right|\le K'_1|x-y|^{\gamma};\tag{11}$$

$$\sum_{2l-k<m}\left|\frac{\partial^l v(t,x)}{\partial x^k\partial t^{l-k}}\right|\le K'_2,\tag{12}$$

where K_1' and K_2' depend only on the constants K_1, K_2, T, m, γ.

Remark. Analogous theorems also hold in the case when $x = (x_1, \dots, x_n)$, $\alpha = (\alpha_1, \dots, \alpha_\nu)$. We omit the formulations of these theorems.

Let us outline the idea of the proof of the theorems.

Consider the equation:

$$x_t^\alpha = \alpha + \int_0^t b(x_u^\alpha)\, d\xi_u \tag{13}$$

(for convenience of exposition we omit the parameter $\omega \in \Omega$).

Formally differentiating (13) with respect to α, for $y_t^\alpha = \partial x_t^\alpha / \partial\alpha$, we obtain the equation

$$y_t = 1 + \int_0^t b'(x_u^\alpha) y_u^\alpha \, d\xi_u, \qquad b'(x) = \frac{db(x)}{dx}. \tag{14}$$

Next let $y_t^{\alpha\beta} = (x_t^\alpha - x_t^\beta)/(\alpha - \beta)$; then for any $\alpha, \beta, \alpha', \beta'$ we have

$$y_t^{\alpha\beta} - y_t^{\alpha'\beta'} = \int_0^t \frac{b(x_u^\alpha) - b(x_u^\beta)}{x_u^\alpha - x_u^\beta} \left(y_u^{\alpha\beta} - y_u^{\alpha'\beta'} \right) d\xi_u +$$

$$+ \int_0^t \left[\frac{b(x_u^\alpha) - b(x_u^\beta)}{x_u^\alpha - x_u^\beta} - \frac{b(x_u^{\alpha'}) - b(x_u^{\beta'})}{x_u^{\alpha'} - x_u^{\beta'}} \right] y_u^{\alpha'\beta'}\, d\xi_u. \tag{15}$$

From this equation, using K. Itô' s formula for stochastic integrals (7), one can obtain the inequalities:

$$\mathbf{M} \left| y_t^{\alpha\beta} - y_t^{\alpha'\beta'} \right|^{2l} \le L' \left(|\alpha - \alpha'|^2 + |\beta - \beta'|^2 \right)^{\gamma l}, \tag{16}$$

$$\mathbf{M} \left| y_t^{\alpha\beta} - y_t^\alpha \right|^{2l} \le L'' |\alpha - \beta|^{2l\gamma}. \tag{17}$$

From inequality (16) (analogously to Kolmogorov' s theorem, see (5), p. 576) it is proved that $y_t^{\alpha\beta}$ is continuous with probability 1 in α, β, and from (17) it follows that

$$\mathbf{P} \left\{ \omega : \lim_{\beta \to \alpha} \frac{x_t^\alpha - x_t^\beta}{\alpha - \beta} = y_t^\alpha \right\} = 1.$$

Next one considers $z_t^{\alpha\beta} = (y_t^\alpha - y_t^\beta)/|\alpha - \beta|^\mu$, $\mu < \gamma$, and proves that

$$\mathbf{M} \left| z_t^{\alpha\beta} - z_{t'}^{\alpha'\beta'} \right|^{2l} \le C \left[(\alpha - \alpha')^2 + (\beta - \beta')^2 + (t - t')^2 \right]^{\lambda l/2}, \tag{18}$$

where $\lambda = \min(\gamma^2, \gamma - \mu) > 0$. From (18) it follows that $z_t^{\alpha\beta}$ is a continuous function in t, α, β for almost all ω, and for any bounded domain G of variation of the variables t, α, β

$$\zeta_G(\omega) = \sup_G |z_t^{\alpha\beta}| < \infty \quad \text{a.s.} \tag{19}$$

For equation (13) in the case where $m = 1$, $\gamma > 0$, Theorems 1 and 2 follow from (16), (17), and (19), if only $|b(x)| + |b'(x)| \leq L_2$ and $|b'(x) - b'(y)| \leq L_1|x - y|^\gamma$. If, however, $b'(x)$ is such that $|b'(x) - b'(y)| \leq L(r)|x - y|^\gamma$ for all $x, y \in [-r, r]$ and is otherwise arbitrary, then Theorem 2 for equation (13) is proved as follows: let $b_r(x)$ coincide with $b(x)$ for $|x| \leq r$, be equal to zero for $|x| \geq r + 1$, and let $|b_r'(x) - b_r'(y)| \leq L'(r)|x - y|^\gamma$ for all $x, y \in \mathbb{R}$, and

$$x_t^\alpha(r) = a + \int_0^t b_r(x_u^\alpha(r))\, d\xi_u. \tag{20}$$

Generalizing Lemma 5 from $^{(1)}$, one can prove that there exists a nonnegative random variable $r_k(\omega) < \infty$ a.s. such that $x_t^\alpha = x_t^\alpha(r)$ for $r \geq r_k(\omega)$ for all $t \in [0, T]$ and $\alpha \in [-k, k]$ simultaneously almost surely. But since for $x_t^\alpha(r)$ the theorem has already been proved and $r_k(\omega) < \infty$ a.s., the theorem is thereby proved also for x_t^α.

From the connection with equation (1) of the corresponding diffusion process and Theorem 1, Theorem 3 follows.

Finally, let us note that Theorems 1 and 3 can be strengthened by allowing functions $a(t, x)$, $b(t, x)$ that grow no faster than $|x|$, but with higher-order derivatives growing only no faster than $|x|^N$ for some $N < \infty$. It is sufficient to require of $\psi(x)$ that $\psi(x)e^{-\theta|x|} \to 0$ as $|x| \to \infty$ for every $\theta > 0$.

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Note: Figure translations are in progress. See original paper for figures.