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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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SOME PROBLEMS OF CYLINDRICAL BENDING OF THREE-LAYER PLATES WITH ALLOWANCE FOR HIGH-ELASTIC DEFORMATION OF FIBERGLASS SKINS

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Analysis of works on the calculation of elastic three-layer plates with a light core ([1,5,6]) and comparison with experiment show that the total deflection of the plate and the stresses in the skins, determined by different theories, turn out to be practically identical, provided only that the parameter θ_2 ([6]), characterizing compression of the core in the transverse direction, is sufficiently large.

Fig. 1

At the same time, in a number of practically important cases, especially for fiberglass skins, in order to determine the load-carrying capacity of plates it is necessary to take into account the inelastic deformations of the skin material, which is the subject of the present work. In these materials, as shown by the authors ([3]), owing to the presence of polymer binders, together with the elastic deformation ε there also occurs a high-elastic deformation ε^*, i.e., a reversible deformation, but one not in phase with the stress. The latter predetermines a number of peculiarities in the behavior of fiberglass plastics: the presence of a nonlinear segment in the tension–compression diagram; a dependence of the mechanical parameters, much greater than in metals, on the rate of deformation and on temperature; the possibility of creep at normal temperature, etc.

The relation between the deformed and stressed states of an anisotropic polymer plate is approximately described by the constitutive equation derived in ([9]) on the basis of the Maxwell–Gurevich equations ([2]) and experimentally confirmed in the uniaxial case ([3,4,9]).

For cylindrical bending (Fig. 1), for the modulus of elasticity of the core in the x-direction the same assumption is adopted as in ([5]), $E_{xx} \to 0$, and for the outer layers—the Reissner assumption ([11]) on the bending stiffness of the outer layers, $B \to 0$. Further, for the skins the components of elastic and high-elastic deformation are taken into account, while for the core only the elastic components are considered.

In the core, from ([5]), for the stresses and displacements we have:

$$\sigma_x = 0, \qquad \sigma_z = -z\,\partial\tau/\partial x + E_{zz}\eta/h, \qquad \tau = \tau(x,t); \tag{1}$$

$$u = -\frac{1}{6E_{zz}}(3zh^2 - z^3)\frac{\partial^2\tau}{\partial x^2} + \frac{1}{G_{xz}}z\tau + \frac{1}{2h}(h^2 - z^2)\frac{\partial\eta}{\partial x} - z\frac{\partial\xi}{\partial x} + \psi. \tag{2}$$

Here:

$$2\eta = w(h) - w(-h), \qquad 2\xi = w(h) + w(-h), \qquad 2\psi = u(h) + u(-h);$$

η, ξ, τ are arbitrary functions of integration, depending on x and t (time). For hinged support (Fig. 1) (see ([6])) $\psi = 0$.

Equilibrium of a small element of the outer layers and the static conditions on the contact surfaces with the core give

$$\partial N_{1,2}/\partial x = \pm\tau b, \qquad \sigma_z(h) = q_1/b, \qquad \sigma_z(-h) = -q_2/b. \tag{3}$$

From the equilibrium condition of the whole three-layer plate as a whole we have

$$\tau = \frac{Q(x,t)}{2hb} = \frac{1}{2hb}\frac{\partial M(x,t)}{\partial x}, \qquad (\bar{\sigma}_x)_{1,2}F = N_{1,2} = \pm\frac{M(x,t)}{2h}. \tag{4}$$

Here Q, M are the transverse force and bending moment acting on the plate as a whole. Thus, in the approximation under consideration, the stresses in the facing and in the core are statically determinate.

At the boundary of the core with the outer layers, using the strain-compatibility conditions, one can write the following kinematic condition:

$$\frac{\partial^2 u(h)}{\partial x\,\partial t} - \frac{\partial^2 u(-h)}{\partial x\,\partial t} = \frac{\partial\varepsilon_{x,1}}{\partial t} - \frac{\partial\varepsilon_{x,2}}{\partial t} \tag{5}$$

and, by virtue of the static conditions (3),

$$\sigma_z(h) + \sigma_z(-h) = (q_1 - q_2)/b, \qquad \sigma_z(h) - \sigma_z(-h) = (q_1 + q_2)/b. \tag{6}$$

The expression for the strain ε_x entering (5) is given by the above-mentioned constitutive equation ([9]), which, together with the equations presented above, forms a complete system with respect to the functions $\xi, \eta, \tau, \varepsilon_x, \sigma_z$. Eliminating $u, \sigma_z, \bar{\sigma}_x$, we arrive at the following basic equations of the problem.

The **constitutive equation** for a uniaxial state of stress in the case of an isothermal process ([9]) here takes the form

$$\frac{1}{h}\frac{\partial \varepsilon_x}{\partial t} = \frac{1}{2\bar{B}}\frac{\partial M}{\partial t} + \frac{\overset{*}{f}\,\bar{m}_x}{2\bar{B}}\,\frac{\bar{E}_x}{\eta_x}\exp|f^*|, \tag{7}$$

where

$$\varepsilon_{x,1} = -\varepsilon_{x,2} = \varepsilon_x, \qquad \varepsilon_x = e_x + \overset{*}{\varepsilon}_x, \qquad e_x = M/2hF\bar{E}_x, \qquad f^* = f/\bar{m}_x, \qquad f = M - 2hFE_{\infty,x}\overset{*}{\varepsilon}_x = M(1+E$$

$\bar{m}_x = 2hFm_x$, $\bar{E}_x$ is the modulus of elasticity, $E_{\infty,x}$ is the modulus of high-elastic strain, η_x is the viscosity coefficient, and m_x is the rate modulus of high-elastic strain of the facing material. All parameters depend on the directions of the axes of elastic symmetry of the facing, and the last two also on the temperature.

On the basis of (5), (2), (7), and (4), putting $x = \xi l$ $(\xi = x/l)$, we obtain

$$\frac{\partial^3 \xi}{\partial \xi^2 \partial t} = -\frac{l^2}{2\bar{B}}\frac{\partial M}{\partial t} + \frac{1}{G_{xz}\,2hb}\frac{\partial^3 M}{\partial \xi^2 \partial t} - \frac{1}{6E_{zz}l^2}\frac{h}{b}\frac{\partial^5 M}{\partial \xi^4 \partial t} - \frac{\overset{*}{f}\,\bar{m}_x l^2}{2\bar{B}}\,\frac{\bar{E}_x}{\eta_x}\exp|f^*|, \tag{8}$$

where $2\bar{B} = 2h^2F\bar{E}_x$ is the bending stiffness of the hollow beam as a whole.

The two basic differential equations (7) and (8) serve to determine the two functions $\varepsilon_x(\xi, t)$ and $\xi(\xi, t)$.

Equation (8) replaces the "beam" equation of an elastic line. The first three terms on the right-hand side characterize, respectively, the elastic strains of the facing and the core, and the last takes into account the influence of the high-elastic strain of the facing. From (6) and (1)

$$\eta = (q_1 - q_2)h/2bE_{zz}. \tag{9}$$

The convergence of the facings does not depend on their strains.

According to Fig. 1 we have the following boundary and initial conditions: for ξ and ε_x

$$\text{at } \xi = 1, \quad \xi = 0; \qquad \text{at } \xi = 0, \quad \partial\xi/\partial\xi = 0; \qquad \text{at } t = 0, \quad \xi = \xi_0,$$

Fig. 2

Figure 2: Fig. 2

$$\varepsilon_x = \varepsilon_{x,0}. \tag{10}$$

The equations obtained are applied to two important regimes: loading with a constant rate of load increase $dq^*/dt = v_q^* =$ const and loading by a constant load $q^* = (q_1^* + q_2^*) =$ const (creep). The quantity $q^* = ql^2/2m_x$ is the dimensionless load.

For the first regime $v_q^* =$ const, in the particular case $E_{\infty,x} \to 0$, integrating equations (7), (8) with allowance for (10) and (10′), we find

$$\varepsilon_x/h = \frac{q^*\bar{m}_x(1-\xi^2)}{2\bar{B}} + \frac{\bar{E}_x\bar{m}_x}{2\bar{B}\eta_x v_q^*}\left\{q^*\exp[q^*(1-\xi^2)] + \frac{1-\exp[q^*(1-\xi^2)]}{1-\xi^2}\right\},$$

$$\zeta = \zeta_{\text{el}} + \left(\overline{E}_x\overline{m}_x l^2 \exp q^*/2\overline{B}\eta_x v_q\right)\left\{(1/2)!\sqrt{\pi q^*}\left[\Phi\left(\sqrt{q^*}\right) - \xi\Phi\left(\xi\sqrt{q^*}\right) + \right.\right. \tag{11}$$

$$\left.\left. + \exp(-q^*)\left[1-\exp q^*(1-\xi^2)\right]/\sqrt{\pi}\sqrt[4]{q^*}\right] - \gamma(\xi, q^*)\right\},$$

where

$$\gamma_1(\xi, q^*) = \exp(-q^*)\int_0^1\int_0^\xi \frac{\left[\exp q^*(1-\xi^2)-1\right]}{(1-\xi^2)}\,d\xi\,d\xi,$$

$\Phi(\xi\sqrt{q^*})$ is the probability integral. Here and below

$$\zeta_{\text{el}} = (5/12)q^*\overline{m}_x l^2(1-6\xi^2/5+\xi^4/5)/2\overline{B} + q^*\overline{m}_x(1-\xi^2)/G_{xz}2hb.$$

The function $\gamma(\xi, q^*)$ is found numerically.

Fig. 2

The solution obtained for $E_{\infty,x} \to 0$ formally corresponds to allowing for residual deformations. As the load increases with time, the shape of the "elastic" line changes. However, analysis shows that this change is insignificant and the shape of the middle line of the plate in the regime under consideration is close to the elastic one for the load at the given instant of time.

For the same regime, in the case $E_{\infty,x} \neq 0$, we obtain

$$\frac{\partial f^*}{\partial q^*} = (1-\xi^2)\{1-\varphi\exp[-|f^*_{\max}|(1-|\varphi|)]\}, \tag{12}$$

$$\zeta = \zeta_{\text{el}} + (\overline{E}_x\overline{m}_x l^2/2\overline{B}E_{\infty,x})\{\Phi^*(\xi,q^*) - \Phi^*(\xi,0) + \\ +(5/12)q^*\overline{m}_x l^2(1-6\xi^2/5+\xi^4/5)\}.$$

Here the first equation and the function $\Phi^*(\xi, q^*)$ are integrated numerically,

$$\Phi^*(\xi,q^*) = \int_1^\xi\int_0^\xi f^*\,d\xi\,d\xi; \qquad \varphi = f^*/f^*_{\max}, \qquad f_{\max} \simeq M_p(1+E_{\infty,x}/\overline{E}_x) - 2hFE_{\infty,x}\varepsilon_{x,p}.$$

M_p; $\varepsilon_{x,p}$ are the bending moment and the deformation at failure.

For $E_{\infty,x} \neq 0$, the shape of the middle line differs even less from the elastic one; therefore, as a first approximation for the regime under consideration, one may assume that the highly elastic deformations in the facing practically do not lead to a change of shape (as a function of ξ).

For the second regime, when $q = \text{const}$, equations (7) and (8) are written as

$$\frac{1}{h}\frac{\partial\varepsilon_x}{\partial t} = \frac{f^*\overline{m}_x}{2\overline{B}}\frac{\overline{E}_x}{\eta_x}\exp|f^*|, \qquad \frac{\partial^3\zeta}{\partial\xi^2\partial t} = -\frac{l^2}{h}\frac{\partial\varepsilon_x}{\partial t}.$$

The integral of the first equation, analogously to (7), taking (10) into account, will be

$$t = [-\operatorname{Ei}(-f^*) + \operatorname{Ei}(-f^*_0)](\eta_x/E_{\infty,x}), \tag{13}$$

where $\operatorname{Ei}(-x)$ is the integral exponential function; for the displacement we have

$$\zeta = \zeta_0 + (\overline{m}_x\overline{E}_x l^2/2\overline{B}E_{\infty,x})\int_1^\xi\int_0^\xi (f^* - f^*_0)\,d\xi\,d\xi \tag{14}$$

for $t = 0$,

$$f = f_0 = M(1+E_{\infty,x}/\overline{E}_x) - 2hF_{\infty,x}\varepsilon_{x,0}.$$

The initial values $\varepsilon_{x,0}$ and ζ_0 depend on the loading prehistory. On the basis of analysis of the first regime, these quantities may be taken approximately as corresponding to the elastic scheme.

In the case $f^* \gg 1$, the integral exponential functions may be replaced by the first term of the asymptotic expansion; then instead of (13)

$$t = [(1/f^*) \exp(-f^*) - (1/f_0^*) \exp(-f_0^*)](\eta_x / E_{\infty,x}). \tag{15}$$

Here two limiting cases are possible. In the first, $f \to M$, i.e., $E_{\infty,x}\varepsilon_x^* \ll 1$. In this case, as follows from (8), the rate of increase of the plate deflection is constant, while the expressions for ζ and ε_x from (13) and (14), or (7) and (8), will be

$$\begin{gathered} \varepsilon_x/h = \varepsilon_{x,0}/h + t\,[\bar{E}_x \bar{m}_x q^*(1-\xi^2)/2\bar{B}\eta_x]\,\exp q^*(1-\xi^2); \\ \zeta = \zeta_0 + t\,[\bar{E}_x \bar{m}_x l^2 \exp q^*/2\bar{B}\eta_x]\{(q^* - {}^1/{}_2)(\sqrt{\pi}/2\sqrt{q^*})[\Phi(\sqrt{q^*}) - \\ -\xi\Phi(\xi\sqrt{q^*}) + (1/\sqrt{\pi}\sqrt[4]{q^*})\exp(-q^*)[1 - \exp q^*(1-\xi^2)]] + \\ +({}^1/{}_{4q^*})[\exp(-q^*\xi^2) - \exp(-q^*)]\}. \end{gathered} \tag{16}$$

This case formally corresponds to creep in the presence of residual deformation ([9]).

The first of equations (16) makes it possible to determine the long-term strength of the structure from the condition that the maximum deformation in the section $\xi = 0$ reaches the value corresponding to the moment of failure, since this quantity for the given direction may be taken as a material parameter ([8]). The formula thus obtained, $t_{\text{pred}} = [(\varepsilon_p - \varepsilon_{x,0})\eta_x/\bar{\sigma}_x]\exp(-\bar{\sigma}_x/m_x)$, is structurally similar to S. N. Zhurkov' s formula ([12]) for an isotherm.

With a stable structure of the facing material, a second limiting case is possible, when $f \to 0$ and the deformation asymptotically approaches the value $\varepsilon_{\text{pred}} \ll \varepsilon_p$. In this case the deflection and deformation of the outer layer will be as follows:

$$\begin{gathered} \zeta = \zeta_{\text{upr}} + ({}^5/{}_{12})(\bar{E}_x \bar{m}_x q^* l^2/2\bar{B}E_{\infty,x})(1 - 6\xi^2/5 + \xi^4/5), \\ \varepsilon_x/h = M(1 + E_{\infty,x}/\bar{E}_x)/2h^2 F E_{\infty,x}. \end{gathered} \tag{17}$$

If the parameter θ_1 ([6]), characterizing the shear of the core, is large ($\theta_1 \to \infty$), then instead of (17) we obtain $\zeta = \bar{\zeta}_{\text{upr}}(1 + \bar{E}_x/E_{\infty,x})$, $\bar{\zeta}_{\text{upr}} = (\zeta_{\text{upr}})_{\theta_1 \to \infty}$, and, consequently, the influence of the highly elastic deformation of the facing on the limiting deflection is expressed in the appearance of a constant multiplier.

Figure 2 shows the shapes, obtained as a result of calculations, of the deflected line of a three-layer plate whose outer layers are made of "equal-strength" SVAM and whose core is made of reinforced foam plastic. Curve 3 represents the "elastic" line at the end of the first loading regime—before the onset of creep. Curves 1, 2 are deflected lines in the creep process, calculated on the basis of (16), when $E_{\infty,x} \to 0$; curve 4 is by (14), (15), when $E_{\infty,x} \simeq \bar{E}_x$. Curves 1 and 4 were calculated for $\varepsilon_{x,\max} \simeq 20\%$, and curve 2 for $\varepsilon_{x,\max} \simeq 2\%$. These curves were determined for a constant load $q^* = 17.257$, but for different instants of time. For comparison, lines 5, 6, and 7 are plotted with dashed lines, calculated according to the elastic scheme ([6]), but for different fictitious loads determined from the condition of equality of the maximum deflections of the corresponding curves. It is seen from the comparison that, when $E_{\infty,x} \simeq \bar{E}_x$, for the considered deformation values the deflected creep line is practically similar to the elastic one, differing only by a time multiplier. When, however, the modulus $E_{\infty,x}$ is very small, then for sufficiently large $\varepsilon_{x,\max}$ the difference in the shape of the elastic line as a function of ξ becomes significant.

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Note: Figure translations are in progress. See original paper for figures.