Mathematics

1963

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Abstract

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Mathematics

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On the embedding of the space of s-smooth functions of n variables into a space of sufficiently smooth functions of a smaller number of variables

(Presented by Academician A. N. Kolmogorov on 14 V 1963)

In the present note we consider the question of the existence of an isomorphic embedding of the space of continuous functions of n variables into a space of continuous functions of a smaller number of variables, under which functions of fixed smoothness from one space are mapped to sufficiently smooth functions from the other space. (By an isomorphism of a Banach space E_1 into a Banach space E_2 we mean a one-to-one continuous linear mapping of the space E_1 onto some closed linear subspace of E_2.)

Let I^n be the n-dimensional cube in the n-dimensional Euclidean space R^n, defined by the inequalities $|x_i| \leq 1$ $(i = 1, 2, \dots, n)$. Denote by $C(I^n)$ the space of all continuous real-valued (or complex-valued) functions defined on the cube I^n, with norm

$$\|f(x)\| = \sup_{x \in I^n} |f(x)|.$$

By $C^{(s)}(I^n)$ we denote the space of all s-times continuously differentiable real-valued (or complex-valued) functions defined on the cube I^n, with norm

$$\|f(x)\|_s = \sum_{k_1+k_2+\dots+k_n \leq s} \sup_{x \in I^n} \left| \frac{\partial^{k_1+k_2+\dots+k_n} f(x)}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}} \right|.$$

Let $n > m$. Our main theorem is the following.

Theorem 1. *If*

$$s > \left\langle \frac{n}{m} \right\rangle \left(1 + \frac{1}{2} \left\langle \frac{n}{m} \right\rangle \right) p,$$

then there exists an isomorphism

$$T : C(I^n) \to C(I^m)$$

having the property

$$T\left[C^{(s)}(I^n)\right] \subset C^{(p)}(I^m)$$

(by $\langle n/m \rangle$ is denoted the integer nearest to n/m from the right).

For convenience of notation we shall assume that the functions take complex values. We shall need one assertion based on Whitney' s results. Denote by $C_0^{(s)}(I_\pi^n)$ the space of all s-times continuously differentiable functions in R^n that vanish outside the cube I^n, defined by the inequalities $|x_i| \le \pi$ $(i = 1, 2, \dots, n)$.

Lemma 1. *There exists a linear continuous operator*

$$M : C(I^n) \to C(I_\pi^n)$$

having the following properties: a)

$$(Mf)(x) \equiv f(x), \qquad x \in I^n$$

for all $f \in C(I^n)$; b)

$$M[C^{(s)}(I^n)] \subset C_0^{(s)}(I_\pi^n).$$

The proof of Lemma 1, formulated in another form, can be found in ([2]).

Let $f(x) \in C(I^n)$ and let M be the extension operator indicated in Lemma 1. Expand the function $(Mf)(x)$ in a Fourier series:

$$(Mf)(x) \sim \sum_{\nu_1,\dots,\nu_n=-\infty}^{\infty} c_{\nu_1,\dots,\nu_n}(f) e^{i[\nu_1 x_1 + \dots + \nu_{nx} n]}, \qquad \text{where } x = (x_1, x_2, \dots, x_n) \in I_\pi^n.$$

Lemma 2. There exists a constant λ_1 such that

$$\left(\sum_{\nu_1,\dots,\nu_n=-\infty}^{\infty} c^2_{\nu_1,\dots,\nu_n}(f) \right)^{1/2} \le \lambda_1 \|f(x)\|, \quad \text{for all } f(x) \in C(I^n).$$

To prove Lemma 2 one must use the continuity of the operator M.

Number the totality of all sets $\{\nu_1, \nu_2, \dots, \nu_n\}$. The set $\{0, 0, \dots, 0\}$ receives number 1. Further, if all sets satisfying the condition $\max_{1\le i\le n} |\nu_i| < \mu$, where μ is a natural number, have been numbered, we number the sets satisfying the condition $\max_{1\le i\le n} |\nu_i| = \mu$. The set that has received number k will be denoted by $\{\nu_1(k), \dots, \nu_n(k)\}$. Put

$$c_k(f) = c_{\nu_1(k),\dots,\nu_n(k)}(f).$$

Lemma 3. If $f(x) \in C^{(s)}(I^n)$, then the sequence $\{k^{s/n} c_k(f)\} = \{\gamma_k\} \in l_2$, i.e.

$$\sum_{k=1}^{\infty} \gamma_k^2 < \infty.$$

Proof. By Lemma 1, $(Mf)(x) \in C_0^{(s)}(I_\pi^n)$. Hence, as is known, it follows that the sequence

$$\left\{ \left[\sum_{i=1}^{n} \nu_i^s(k) \right] c_k(f) \right\} \in l_2.$$

By the numbering, for any number k one of the numbers $\nu_i(k)$ $(i = 1, 2, \dots, n)$ is not less than $\frac{1}{2}(k^{1/n} - 1)$. Therefore the sequence $\{k^{s/n} c_k(f)\} \in l_2$.

The following assertion is known:

Lemma 4. There exists a continuous mapping of the segment I onto the cube I^n,

$$x_i = x_i(t) \quad (i = 1, 2, \dots, n),$$

such that the functions $x_i(t)$ satisfy the Hölder condition with exponent $1/n$ and constant λ:

$$|x_i(t+\delta) - x_i(\delta)| \le \lambda \delta^{1/n} \quad (i = 1, 2, \dots, n).$$

Lemma 5. Put

$$\varphi_k(t) = e^{i[\nu_1(k)x_1(t) + \dots + \nu_n(k)x_n(t)]},$$

where $x_i(t)$ $(i = 1, 2, \dots, n)$ is the mapping indicated in Lemma 4. Then

$$|\varphi_k(t+\delta) - \varphi_k(t)| \le n\lambda(k\delta)^{1/n}, \quad t \in I \quad (k = 1, 2, \dots).$$

Proof.

$$|\varphi_k(t+\delta) - \varphi_k(t)| \le [\nu_1(k) + \dots + \nu_n(k)]\lambda\delta^{1/n} \le nk^{1/n}\lambda\delta^{1/n},$$

where the first estimate is obtained by using Lemma 4, and the second estimate follows from the fact that, by the numbering, for any number k we have

$$|\nu_i(k)|^n \le k \quad (i = 1, 2, \dots, n).$$

We smooth the functions $\varphi_k(t)$ by averaging:

$$\widetilde{\varphi}_k(t) = \frac{1}{(2\delta_k)^p} \int_{|t-t_p| \le \delta_k} dt_p \int_{|t_p - t_{p-1}| \le \delta_k} dt_{p-1} \cdots \int_{|t_2 - t_1| \le \delta_k} \varphi_k(t_1)\, dt_1. \quad (1)$$

Lemma 6. The inequality holds

$$\|\varphi_k(t) - \widetilde{\varphi}_k(t)\| \le n\lambda(pk\delta_k)^{1/n} \quad (k = 1, 2, \dots).$$

Lemma 7. The inequality is valid

$$\|\varphi_k^{(p)}(t)\| \le \frac{n\lambda}{2} \frac{(2k\delta_k)^{1/n}}{\delta_k^p} \quad (k = 1, 2, \dots).$$

Proof. Lemma 6 follows directly from Lemma 5. To prove Lemma 7 one must differentiate expression (1) p times and use Lemma 5.

We shall present the proof of Theorem 1 in the case where $m = 1$ (the case of arbitrary m causes no additional difficulties). The desired isomorthe morphism $T : C(I^n) \to C(I)$ is constructed by the formula

$$T[f(x_1, x_2, \dots, x_n)] =$$

$$= f(x_1(t), x_2(t), \dots, x_n(t)) + \sum_{k=1}^{\infty} c_k(f)\,[\widetilde{\varphi}_k(t) - \varphi_k(t)]. \tag{2}$$

In order that the series on the right-hand side of (2) converge uniformly, it is sufficient that the inequality

$$\sum_{k=1}^{\infty} \|\widetilde{\varphi}_k(t) - \varphi_k(t)\|^2 < \infty$$

hold. To this end we fix $0 < \varepsilon < 1$ and put $\delta_k = \lambda_2 k^{-s/np}$, where the constant λ_2 is chosen from the condition

$$\sum_{k=1}^{\infty} \|\widetilde{\varphi}_k(t) - \varphi_k(t)\|^2 < \left(\frac{\varepsilon}{\lambda_1}\right)^2. \tag{3}$$

This can be done, since, by Lemma 6,

$$\sum_{k=1}^{\infty} \|\widetilde{\varphi}_k(t) - \varphi_k(t)\|^2 \le p^{2/n}(n\lambda)^2 \sum_{k=1}^{\infty} (k\delta_k)^{2/n} = (n\lambda)^2(\lambda_2 p)^{2/n} \sum_{k=1}^{\infty} k^{2/n - 2s/(n^2 p)} < \infty. \tag{4}$$

The last inequality is obvious if one recalls that $s > (n + n^2/2)p$. Using the Cauchy-Bunyakovsky inequality, Lemma 2, and inequality (3), we have

$$\left\|\sum_{k=1}^{\infty} c_k(f)(\widetilde{\varphi}_k - \varphi_k)\right\| \le \left(\sum_{k=1}^{\infty} c_k^2(f)\right)^{1/2} \left(\sum_{k=1}^{\infty} \|\widetilde{\varphi}_k - \varphi_k\|^2\right)^{1/2} \le \varepsilon\|f\|.$$

From the last inequality and formula (2) we obtain

$$(1 - \varepsilon)\|f(x_1, \dots, x_n)\| \le \|T[f(x_1, x_2, \dots, x_n)]\| \le$$

$$\le (1 + \varepsilon)\|f(x_1, x_2, \dots, x_n)\|. \tag{5}$$

The operator T is linear by virtue of formula (2) and the linearity of the operator M. The operator T is continuous and one-to-one by virtue of (5). Consequently,

T is an isomorphism of $M(I^n)$ into $C(I)$. It remains to verify that $T[C^{(s)}(I^n)] \subset C^{(p)}(I)$. If $f(x) \in C^{(s)}(I^n)$, then the Fourier series of the function $(Mf)(x)$ converges to it uniformly. Consequently,

$$f(x_1(t), x_2(t), \dots, x_n(t)) = \sum_{k=1}^{\infty} c_k(f)\varphi_k(t),$$

where convergence is understood in the sense of the norm of the space $C(I)$. Therefore formula (2) for $f(x) \in C^{(s)}(I^n)$ takes the form

$$T[f(x_1, x_2, \dots, x_n)] = \sum_{k=1}^{\infty} c_k(f)\widetilde{\varphi}_k(t) = \tilde{f}(t).$$

The function $\tilde{f}(t)$, evidently, will have p continuous derivatives if we prove that the series

$$\sum_{k=1}^{\infty} c_k(f)\widetilde{\varphi}_k^{(p)}(t)$$

converges uniformly. Using Lemma 3, Lemma 7, the Cauchy–Bunyakovsky inequality, and inequality 4, we have

$$\sum_{k=1}^{\infty} |c_k(f)| \cdot \|\widetilde{\varphi}_k^{(p)}(t)\| \leq \frac{n\lambda}{2} \sum_{k=1}^{\infty} \frac{\gamma_k (2k\delta_k)^{1/n}}{k^{s/n}\delta_k^p} =$$

$$= \frac{n\lambda}{2\lambda_2^p} \sum_{k=1}^{\infty} \gamma_k (2k\delta_k)^{1/n} \leq \frac{n\lambda}{2\lambda_2^p} \left(\sum_{k=1}^{\infty} \gamma_k^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} (2k\delta_k)^{2/n} \right)^{1/2} < \infty.$$

Consequently, $\tilde{f}(t) \in C^{(p)}(I)$. The theorem is proved.

Remark. Let us note that for any $0 < \varepsilon < 1$ the desired isomorphism $T : C(I^n) \to C(I^m)$ can be constructed in the form of an ε-isometric isomorphism:

$$(1-\varepsilon)\|f\| \leqslant \|Tf\| \leqslant (1+\varepsilon)\|f\|; \qquad f \in C(I^n); \quad Tf \in C(I^m).$$

It is not difficult to prove that there does not exist an isometric isomorphism $T : C(I^n) \to C(I^m)$ having the property $T[C^{(s)}(I^n)] \subset C^{(p)}(I^m)$ for any s, p and $n > m$.

Theorem 2. *If $s < \dfrac{n}{m}p$, then there does not exist an isomorphism $T : C(I^n) \to C(I^m)$ having the property*

$$T[C^{(s)}(I^n)] \subset C^{(p)}(I^m).$$

This theorem, as we shall now see, is a simple consequence of a theorem of A. N. Kolmogorov (1). The scheme of the argument is borrowed from the works of A. A. Milyutin. Let K^n be the unit ball of the space $C(I^n)$, and let K_s^n be the unit ball of the space $C^{(s)}(I^n)$. Let $N(K_s^n, \varepsilon K^n)$ be equal to the smallest number of translates of the set εK^n by which the set K_s^n can be covered. A. N. Kolmogorov' s theorem asserts that there exist two constants $A_{n,s}$ and $B_{n,s}$, independent of ε, such that

$$B_{n,s}\left(\frac{1}{\varepsilon}\right)^{n/s} \leqslant \log N(K_s^n, \varepsilon K^n) \leqslant A_{n,s}\left(\frac{1}{\varepsilon}\right)^{n/s}.$$

Proof of Theorem 2. Suppose that under the hypotheses of the theorem there exists an isomorphism $T : C(I^n) \to C(I^m)$ having the property $T[C^{(s)}(I^n)] \subset C^{(p)}(I^m)$. Then the mapping $T : C^{(s)}(I^n) \to C^{(p)}(I^m)$ is closed. Indeed, if the sequence $\{g_k\} \in C^{(s)}(I^n)$ and $\|g_k - g\|_s \to 0$, while the sequence $\{Tg_k\} \in C^{(p)}(I^m)$ and $\|Tg_k - \varphi\|_p \to 0$, then $\|g_k - g\| \to 0$ and $\|Tg_k - \varphi\| \to 0$ in $C(I^m)$, since $\|g_k - g\| \leqslant \|g_k - g\|_s$, and $\|Tg_k - \varphi\| \leqslant \|Tg_k - \varphi\|_p$.

From the continuity of the mapping $T : C(I^n) \to C(I^m)$ it follows that $\varphi = Tg$. By the closed graph theorem the mapping $T : C^{(s)}(I^n) \to C^{(p)}(I^m)$ is continuous. Consequently, there exists a constant R_1 such that $T(K_s^n) \subset R_1 K_p^m$. Since $T : C(I^n) \to C(I^m)$ is an isomorphism, there exists a constant R_2, independent of ε, such that

$$N(K_s^n, \varepsilon K^n) \leqslant N(R_2 T(K_s^n), \varepsilon K^m) \leqslant N(R_2 R_1 K_p^m, \varepsilon K^m)$$

for every $\varepsilon > 0$. Hence, by A. N. Kolmogorov' s theorem,

$$B_{n,s}\left(\frac{1}{\varepsilon}\right)^{n/s} \leqslant \log N(K_s^n, \varepsilon K^n) \leqslant \log N(R_2 R_1 K_p^m, \varepsilon K^m) \leqslant A_{m,p}\left(\frac{R_2 R_1}{\varepsilon}\right)^{m/p}$$

for every $\varepsilon > 0$. The latter is impossible, since $n/s > m/p$. The theorem is proved.

The author expresses his deep gratitude to A. G. Vitushkin for posing the problem and for supervising the work.

Received
22 IV 1963

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Note: Figure translations are in progress. See original paper for figures.