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Abstract

Full Text

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ON THE ASYMPTOTIC CONDITION OF LEHMANN, SYMANZIK, AND ZIMMERMANN

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Physics

1. In recent years much attention has been devoted to the "axiomatic" approach to the construction of quantum field theory, based not on equations of motion but on a certain system of basic propositions or "axioms." Such systems, proposed by various authors, differ first of all in the form of the causality condition. Thus, in the widely adopted system of Lehmann, Symanzik, and Zimmermann ([1]) (LSZ), this condition is connected not only with the causality condition named by these authors—the requirement of commutativity of the Heisenberg field operators outside the light cone—but also with the "asymptotic condition" introduced by them, the role of which has been studied in a number of works (see, for example, ([2])). We wish to clarify here to what extent the latter condition follows from the axiomatics chosen in ([3])*, on the basis of the causality condition proposed by N. N. Bogolyubov ([4]).

2. In order to formulate the LSZ asymptotic condition, let us define, for any field, say $B(x)$, depending on four space-time coordinates, the three-dimensional Fourier transforms

$$B_{\mathbf{k}}(x^0) \equiv B_{\mathbf{k}}^{(-)}(x^0) = \frac{-i}{(2\pi)^{3/2}} \int \frac{dx\, e^{ikx}}{\sqrt{2k^0}} \left(ik^0 B(x) - \frac{\partial B(x)}{\partial x^0} \right) \Bigg|_{k^0=\sqrt{\mathbf{k}^2+m^2}} \tag{1}$$

and the complex conjugate $B_{\mathbf{k}}^{+}(x^0) \equiv B_{\mathbf{k}}^{(+)}(x^0)$ (we take $B(x)$ to be Hermitian), depending on time, whose specification fixes $B(x)$. These Fourier transforms are constructed analogously to the creation and annihilation operators of the free field and are normalized in such a way that for the out-field $\varphi(x)$**: $\varphi_{\mathbf{k}}^{(\mp)}(x^0) = a^{(\mp)}(\mathbf{k})$, i.e. $\varphi_{\mathbf{k}}(x^0) = a(\mathbf{k})$ and $\varphi_{\mathbf{k}}^{+}(x^0) = a^{+}(\mathbf{k})$. It is easy to see that, for the Heisenberg field $A(x)$, the operators $A_{\mathbf{k}}, A_{\mathbf{k}}^{+}$ will be, at fixed x^0, the annihilation and creation operators of the free field coinciding, together with the first derivative, with $A(x)$ at the instant x^0 (the tangent field of Yang–Feldman ([5])).

The asymptotic condition is now written in the form of the requirement of weak convergence of the operators $A_{\mathbf{k}}, A_{\mathbf{k}}^{+}$ to the operators $a^{(\mp)}(\mathbf{k})$ and $a_{\text{in}}^{(\mp)}(\mathbf{k})$ as x^0

tends respectively to $+\infty$ or to $-\infty$:

$$\lim_{x^0 \to +\infty} \langle \alpha | A_{\mathbf{k}}^{(\mp)}(x^0) | \beta \rangle = \langle \alpha | a^{(\mp)}(\mathbf{k}) | \beta \rangle, \tag{2^1}$$

$$\lim_{x^0 \to -\infty} \langle \alpha | A_{\mathbf{k}}^{(\mp)}(x^0) | \beta \rangle = \langle \alpha | a_{\text{in}}^{(\mp)}(\mathbf{k}) | \beta \rangle \tag{2^2}$$

for any states $\langle \alpha |$ and $| \beta \rangle$***.

3. In the axiomatics ([3]) there was no need at all for the concept of a Heisenberg field. Therefore, relying on analogy with the usual theory, we have the right to define the Heisenberg field $A(x)$ by the equality

$$A(x) = T(\varphi(x) S) S^{+}, \tag{3}$$

from which a representation of the Yang–Feldman type ([5]) will follow.****

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* We use the notation of ([3]).

** For simplicity we work with one Hermitian scalar field.

*** In distinction from ([1]), we are content with the ordinary Fourier transforms of the continuous spectrum, without resorting to a normalized-to-1 orthogonal system of solutions of the Klein–Gordon equation in a definite volume, which for an infinite region can be defined only by means of a certain limiting process. It is easy to see that such a process, not being connected with crossing the energy surface, would change nothing in the following arguments.

**** The definition (3) and the representation (4) will be discussed in more detail elsewhere.

$$A(x) = \varphi(x) - \int D^{\text{adv}}(x - x')\, j(x')\, dx', \tag{4}$$

which leads, for the operators $A_{\mathbf{k}}$ and $A_{\mathbf{k}}^{+}$, to the expressions

$$A_{\mathbf{k}}^{(\mp)}(x^0) = a^{(\mp)}(\mathbf{k}) \pm \frac{i}{(2\pi)^{3/2}} \int \frac{dx\, e^{\pm ikx}}{\sqrt{2k^0}} \int dx' \left(\pm i k^0 D^{\text{adv}}(x - x') - \frac{\partial D^{\text{adv}}(x - x')}{\partial x^0} \right) j(x'), \quad k^0 = \sqrt{\mathbf{k}^2 + m^2} \tag{5}$$

Let us check whether these operators will satisfy the asymptotic conditions (2). Passing to matrix elements, we shall compute them between l- and s-particle states (cf. ([6]), equation (4)); here it is convenient to use the functions $J((p)_l; (q)_s)$ introduced in ([7]) (the notation, for example, $(p)_l$, means, for brevity, $p_1, \dots, p_l$):

$$\langle (p)_l | j(x') | (q)_s \rangle = e^{iPx'} J((p)_l; (q)_s);$$

$$P = \sum_1^l p - \sum_1^s q; \qquad p_i^0 = \sqrt{\mathbf{p}_i^2 + m^2}, \quad q_j^0 = \sqrt{\mathbf{q}_j^2 + m^2}. \tag{6}$$

We then obtain

$$\langle (p)_l | A_{\mathbf{k}}^{(\mp)}(x^0) - a^{(\mp)}(\mathbf{k}) | (q)_s \rangle =$$

$$= -\frac{(2\pi)^{3/2}}{\sqrt{2k^0}} \, \delta(\mathbf{P} \pm \mathbf{k}) \, e^{i(P^0 \pm k^0)x^0} \, \frac{k^0 \mp P^0}{m^2 - P^2 - i\varepsilon P^0} \, J((p)_l; (q)_s). \tag{7}$$

Using the spatial δ-function appearing in (7) and the fact that the vector $\mathbf{k}$ lies on the energy surface, one can transform the fraction $\dfrac{k^0 \mp P^0}{m^2 - P^2 - i\varepsilon P^0}$ in (7) to the form $\dfrac{1}{k^0 \pm P^0 \pm i\varepsilon}$, and the limits we need are written as

$$\lim_{t \to +\infty} \langle (p)_l | A_{\mathbf{k}}^{\mp}(x^0) - a^{(\mp)}(\mathbf{k}) | (q)_s \rangle = \mp \frac{(2\pi)^{3/2}}{\sqrt{2k^0}} \, \delta(\mathbf{P} \pm \mathbf{k}) \, L_+(P^0 \pm k^0) \, J((p)_l; (q)_s), \tag{8}$$

and the analogous expression for $t \to -\infty$, with $L_+(P^0 \pm k^0)$ replaced by $L_-(P^0 \pm k^0)$, where

$$L_\pm(\alpha) = \lim_{t \to \pm\infty} \frac{e^{i\alpha t}}{\alpha + i\varepsilon}. \tag{9}$$

4. It is clear that the limits (9) do not exist in the ordinary sense. We shall therefore understand them in an improper sense, i.e. in the sense of carrying out the limiting transition after integration with functions from some class of sufficient regularity. Without setting ourselves the aim of finding this class, we note that the limit (9) will in any case exist on the class of functions $f(\alpha)$ analytic in the whole complex α-plane, except for a finite number of poles (possibly located at infinity), and regular on the real axis.

Indeed, a function of such a class can always be represented as the sum of a rational function $\varphi(\alpha)$, decreasing at infinity, and a polynomial $P(\alpha)$. For the first function, the integral $\int \dfrac{e^{i\alpha t}}{\alpha + i\varepsilon} \varphi(\alpha) \, d\alpha$ is computed by means of the residue theorem, and the contributions from all poles will contain exponentials that decay in the limiting transition $t \to \pm\infty$. The only exception is the pole at $\alpha = -i\varepsilon$, which gives the contribution $-2\pi i \varphi(0)$ as $t \to -\infty$ and a zero contribution as $t \to +\infty$. The integral of the polynomial will be equal to the same polynomial in $-i\dfrac{\partial}{\partial t}$, acting on the discontinuous function $-2\pi i\vartheta(-t)$, and therefore, like the first integral, will tend to $-2\pi i P(0)$ as $t \to -\infty$ and to 0 as $t \to +\infty$. Consequently, in the improper sense,

$$L_+(\alpha) = 0, \qquad L_-(\alpha) = -2\pi i\delta(\alpha). \tag{10}$$

5. Returning now to expression (8), we see that

$$\lim_{x^0\to+\infty} \langle (p)_l | A_k^{(\mp)}(x^0) - a^{(\mp)}(k) | (q)_s \rangle = 0, \tag{11^1}$$

$$\lim_{x^0\to-\infty} \langle (p)_l | A_k^{(\mp)}(x^0) - a^{(\mp)}(k) | (q)_s \rangle = \pm \frac{i(2\pi)^{5/2}}{\sqrt{2k^0}}\, \delta(P \pm k)\, J((p)_l; (q)_s). \tag{11^2}$$

From (11^1) it is immediately clear that the asymptotic condition (2^1) is satisfied. To verify the fulfillment of condition (2^2), let us recall that, by definition,

$$a_{\text{in}}^{(\mp)}(k) = S a^{(\mp)}(k) S^+ = a^{(\mp)}(k) - [a^{(\mp)}(k), S] S^+$$

$$= a^{(\mp)}(k) \pm \frac{i}{(2\pi)^{5/2}} \int \frac{dx\, e^{\pm ikx}}{\sqrt{2k^0}}\, j(x), \tag{12}$$

where we have used (2.20) of ([3]), and hence

$$\langle (p)_l | a_{\text{in}}^{(\mp)}(k) - a^{(\mp)}(k) | (q)_s \rangle = \pm i \frac{(2\pi)^{5/2}}{\sqrt{2k^0}}\, \delta(P \pm k)\, J((p)_l; (q)_s). \tag{13}$$

Comparing the right-hand side with the right-hand side of (11^2), we see that

$$\lim_{x^0\to-\infty} \langle (p)_l | A_k^{(\mp)}(x^0) - a_{\text{in}}^{(\mp)}(k) | (q)_s \rangle = 0, \tag{14}$$

i.e., the asymptotic condition is also fulfilled for $x^0 \to -\infty$.

Thus, we have shown that, with sufficient regularity of the matrix elements $J((p)_l; (q)_s)$, considered as functions of $\alpha_\pm = P^0 \pm k^0 = \sum (p^2 + m^2)^{1/2} - \sum (q^2 + m^2)^{1/2} \pm ((\sum p - \sum q)^2 + m^2)^{1/2}$, the LSZ asymptotic condition follows from the system of fundamental postulates ([3]).

6. It is easy to see, however, that the preceding arguments are invalid for the cases $l+s = 1$. Indeed, by virtue of the stability condition for one-particle states

([3]), §4.1, $J(p;-) = J(-;q) = 0$, and the last factor on the right-hand side of (11^2) vanishes. At the same time, however, the time δ-function turns out to be a consequence of the spatial one, and an infinite factor $\delta(0)$ arises. Therefore the case $l + s = 1$ requires a special investigation,* which it is more convenient to carry out not with the matrix elements of the combinations (1), but with the matrix elements of the field $A(x)$ itself.

* Before passing to such an investigation, let us note that since, for example, $J(p;-) = \langle p|j(0)|0\rangle$, it can be expressed, replacing the "amputation" $\langle p|$ by a variational derivative, through the two-particle Green function introduced in ([3]), § 4:

$$J(p;-) = -\frac{1}{(2\pi)^{3/2}\sqrt{2p^0}}\, g^{\mathrm{ret}}(p), \qquad p^0 = \sqrt{\mathbf{p}^2 + m^2},$$

and, using the spectral representation ([3]) (4.38) for the latter, represented (under the assumption that the degree of growth is $n = 1$) in the form

$$J(p;-) = -\frac{C_1(m^2 - p^2)}{(2\pi)^{3/2}\sqrt{2p^0}} + O(m^2 - p^2)^2.$$

It is now clear that, with respect to the substitution of such expressions for the one-particle matrix elements, expression (11) and $\lim_{x^0\to\pm\infty}$ of (7) behave differently. The first, effectively containing the δ-function $\delta(m^2 - p^2)$, immediately vanishes, whereas the second contains the denominator $(m^2 - p^2)$, which, upon cancellation with $(m^2 - p^2)$ in $J(p;-)$, will lead to a finite quantity proportional to C_1. True, such a cancellation contradicts the original LSZ prescriptions for understanding the asymptotic condition in the form of equality of matrix elements. Indeed, taking the matrix element actually corresponds to multiplication by $\delta(m^2 - p^2)$, and we are dealing with the ($\sim C_1$) indeterminate expression

$$\frac{1}{(m^2 - p^2)}(m^2 - p^2)\delta(m^2 - p^2), \tag{*}$$

which depends on the order of the multiplications. The simple result mentioned arises if one first multiplies the first two of the left factors, whereas the LSZ prescription requires primary use of the extreme right one. A more detailed study of expression (*), understood in the sense of a certain limit of some composed* $(m^2 - p^2)^{-1}$ *and* $\delta(m^2 - p^2)$*, shows that it is equal to* $\alpha\delta(m^2 - p^2)$*, where the value of the coefficient* α *depends on the order of the limiting transitions and, under reasonable assumptions, lies within the limits* $0 \le \alpha \le 1$*. Let us also note that, in computing one-particle matrix elements of the in-field in (13), no ambiguities arise, since for it* D^{adv} *in (5) is replaced by* D*, and instead of (*) there appears the expression $\delta(m^2 - p^2)(m^2 - p^2)\delta(m^2 - p^2)$, which is unambiguously equal to 0.

Returning to its definition (3), we have

$$\begin{aligned}\langle\mathbf{p}|A(x)|0\rangle &= \langle\mathbf{p}|T(\varphi(x)S)S^{+}|0\rangle = \\ &= \langle\mathbf{p}|:\varphi(x)S:|0\rangle - i\int dx' D^{c}(x-x')\left\langle\mathbf{p}\left|\frac{\delta S}{\delta\varphi(x')}\right|0\right\rangle = \\ &= \langle\mathbf{p}|\varphi(x)|0\rangle - \frac{i}{(2\pi)^{3/2}}\int dx'\int\frac{dy}{\sqrt{2p^{0}}}e^{ipy}D^{c}(x-x')\left\langle 0\left|\frac{\delta^{2}S}{\delta\varphi(x')\delta\varphi(y)}\right|0\right\rangle,\end{aligned} \tag{15}$$

and an analogous expression for $\langle 0|A(x)|\mathbf{q}\rangle$. Separating here the exponential dependence on the coordinates and using the spectral representation, found in ([3]), § 4, of the second variational derivative, we arrive at

$$\langle\mathbf{p}|A(x)-\varphi(x)|0\rangle = \frac{e^{ipx}}{(2\pi)^{3/2}\sqrt{2p_{0}}}\frac{g^{c}(-p)}{m^{2}-p^{2}-i\varepsilon} = \frac{e^{ipx}}{(2\pi)^{3/2}\sqrt{2p_{0}}}\frac{C_{1}(m^{2}-p^{2})+O(m^{2}-p^{2})^{2}}{(m^{2}-p^{2}-i\varepsilon)}; \tag{16^1}$$

$$\langle 0|A(x)-\varphi(x)|\mathbf{q}\rangle = \frac{e^{-iqx}}{(2\pi)^{3/2}\sqrt{2q^{0}}}\frac{g^{c}(q)}{m^{2}-q^{2}-i\varepsilon} = \frac{e^{-iqx}}{(2\pi)^{3/2}\sqrt{2q^{0}}}\frac{C_{1}(m^{2}-q^{2})+O(m^{2}-q^{2})^{2}}{(m^{2}-q^{2}-i\varepsilon)}, \tag{16^2}$$

i.e., precisely expressions of the type of expression (*) (since $p^2 = q^2 = m^2$!). No other two-valued moments will occur, and the matrix elements participating in the asymptotic condition from combinations (1) will become, independently of the time x^0, equal to

$$\begin{gathered}\langle\mathbf{p}|A_{\mathbf{k}}(x^{0})-a(\mathbf{k})|0\rangle = 0; \qquad \langle\mathbf{p}|A^{+}_{\mathbf{k}}(x^{0})-a^{+}(\mathbf{k})|0\rangle = \alpha C_{1}\delta(\mathbf{p}-\mathbf{k}); \\ \langle 0|A_{\mathbf{k}}(x^{0})-a(\mathbf{k})|\mathbf{q}\rangle = \alpha C_{1}\delta(\mathbf{k}-\mathbf{q}); \qquad \langle 0|A^{+}_{\mathbf{k}}(x^{0})-a^{+}(\mathbf{k})|\mathbf{q}\rangle = 0,\end{gathered} \tag{17}$$

where the coefficient α expresses the dependence on the order of the limiting transitions.

7. Thus, we see that the asymptotic condition in the LSZ form is not a consequence of the system of axioms proposed in ([3]), and, consequently, narrows the admissible class of theories. Since the latter system proved sufficient for establishing connections (of the character of dispersion relations) between observable quantities, there is as yet no visible need for such a narrowing. If, however, this is desired, then the choice of the LSZ condition cannot be considered successful, since it leads to expressions devoid of a clear meaning.* It seems to us that in such a case it would be more rational to require that the coefficient at the pole with the basic mass in the full Green' s function be equal to unity, i.e., that the constant C_1 be equal to zero, whence, as is seen from (17), there would follow the unambiguous fulfillment of the asymptotic condition.

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* We do not touch here on the connection between the system of axioms of ([3]) and the asymptotic condition, considered by Haag ([8]) and Ruelle ([9]), in Wightman' s axiomatics ([10]), which would require a special investigation.

Note: Figure translations are in progress. See original paper for figures.