A SYSTEM OF DEFINING PARAMETERS CHARACTERIZING THE GEOMETRIC PROPERTIES OF AN ANISOTROPIC MEDIUM

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Abstract

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A SYSTEM OF DEFINING PARAMETERS CHARACTERIZING THE GEOMETRIC PROPERTIES OF AN ANISOTROPIC MEDIUM

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Depending on the nature of the continuous medium under consideration and on the character of the physical phenomena taken into account, the state of each particle of the continuous medium may be characterized by certain physical quantities, such as density, temperature, concentrations of the chemical components of the substance, the strain tensor, etc. The geometric characteristics of the internal structure of a particle, such as crystallographic symmetry, etc., are also important.

By a system of defining parameters we understand a set of functionally independent (dimensional or dimensionless) quantities that uniquely determine the state and motion of the medium, its physical and geometric properties (^1). The local state of the medium is considered known if the values of the defining parameters are specified at the given point of the medium.

The anisotropy properties of a continuous medium in a number of cases are connected with the presence, at the point of the medium under consideration, of a bundle of equivalent directions*. The symmetry of the equivalent directions at each point of the medium is characterized by some point group (a subgroup of the full orthogonal group). It is convenient to classify anisotropic media according to their point symmetry groups.

A medium is called **isotropic** if its point symmetry group is either the full orthogonal group or the proper orthogonal group, and **anisotropic** in the remaining cases. A medium is called **crystalline** if its point symmetry group is one of the 32 finite crystallographic groups. We shall call a medium a **texture** if its point symmetry group is one of the 7 continuous (compact) subgroups of the full orthogonal group. In other words, a medium is called a texture if the equivalent directions in each particle of the medium are brought into coincidence under rotation of the particle through any angle about some fixed axis in the particle.

It is possible to consider media or bodies with other point symmetry groups as well; these media or bodies differ from crystals and from textures (^2).

Let us have a representation T^i_j of the point symmetry group $\mathfrak{T}$**. We shall say that the contravariant components of the tensor A are invariant–

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* Two directions in a particle of a continuous medium are equivalent if all the physical properties of the particle are the same along these directions.

** In a curvilinear coordinate system x^i with metric tensor g^{ij}, each element T of the point group $\mathfrak{T}$ has some matrix representation T^i_j, which satisfies the equation

$$g_{ij} T^i_k T^j_l - g_{kl} = 0. \tag{1a}$$

are invariant with respect to the group $\mathfrak{T}$, if

$$T^{i_1}_{j_1} T^{i_2}_{j_2} \ldots T^{i_n}_{j_n} A^{j_1 j_2 \ldots j_n} = A^{i_1 i_2 \ldots i_n} \tag{1}$$

for every element T of the group $\mathfrak{T}$. By virtue of the orthogonality of the group $\mathfrak{T}$, the property of invariance of the components of the tensor A extends to its components with any arrangement of indices, when the manipulation of indices is carried out with the aid of the tensors g_{ij} and g^{ij}, where $g^{i\alpha} g_{\alpha j} = \delta^i_j$; in this case one may speak of the invariance of the tensor A with respect to the group $\mathfrak{T}$. If the tensor A with components $A^{i_1 i_2 \ldots i_n}$ is invariant with respect to the group $\mathfrak{T}$, then it is easy to see that the tensor B with components $A^{\alpha \cdot i_3 \ldots i_n}_{\cdot \alpha}$ is also invariant with respect to the group $\mathfrak{T}$.

In the present note we state the following basic proposition.

Any tensor of arbitrary rank, invariant with respect to a given point group $\mathfrak{T}$, can be represented in the form of a linear combination of tensors constructed, by means of invariant tensor operations (multiplication, contraction, and permutation of indices), from some finite set of tensors $\{A_{(r)}\}$ $(r = 1, 2, \ldots, l)$ such that none of its parts possesses the same property; moreover the tensors $\{A_{(r)}\}$ and their number l depend only on the type of the group $\mathfrak{T}$.

If a complete rational tensor basis, introduced in ([7]), corresponding to the given group $\mathfrak{T}$ is known, then by invariant tensor operations one can construct the tensors $\{A_{(r)}\}$ from the elements of this basis. The elements of a complete rational basis for continuous (compact) point groups and for point groups of crystals of the lower and some middle syngonies are given in ([5]). The existence of a finite set of tensors $\{A_{(r)}\}$ for every finite or compact subgroup of the full orthogonal group follows from the existence, for these subgroups, of a finite set of typical basic polynomial invariants ([6]).

It is obvious that, as a system of defining parameters characterizing the anisotropy of a medium, one may take the elements of the indicated finite set of tensors $\{A_{(r)}\}$ corresponding to the group $\mathfrak{T}$ describing the prescribed symmetry of the medium.

Nonlinear relations for tensors characterizing physical and geometrical properties or regularities in an anisotropic medium are invariant tensor functions of the physical and geometrical defining parameters.

Let a tensor function F of tensor defining parameters $\{A_{(r)}\}$ and $\{B_{(s)}\}$ be given, specifying a physical law:

$$\hat{F}^{i_1 i_2 \dots i_n} = \hat{F}^{i_1 i_2 \dots i_n}\left(\hat{g}_{\alpha\beta},\, \hat{A}^{j_1 j_2 \dots j_{\sigma_r}},\, \hat{B}^{k_1 k_2 \dots k_{\tau_s}}\right), \tag{2}$$

where

$$F = \hat{F}^{i_1 i_2 \dots i_n} \hat{\partial}_{i_1} \hat{\partial}_{i_2} \dots \hat{\partial}_{i_n},$$

$$A_{(r)} = \hat{A}^{j_1 j_2 \dots j_{\sigma_r}} \hat{\partial}_{j_1} \hat{\partial}_{j_2} \dots \hat{\partial}_{j_{\sigma_r}} \qquad (r = 1, 2, \dots, l)$$

are the geometrical defining parameters;

$$B_{(s)} = \hat{B}^{k_1 k_2 \dots k_{\tau_s}} \hat{\partial}_{k_1} \hat{\partial}_{k_2} \dots \hat{\partial}_{k_{\tau_s}} \qquad (s = 1, 2, \dots, m)$$

are the physical defining parameters; $\hat{g}_{\alpha\beta} = (\hat{\partial}_\alpha, \hat{\partial}_\beta)$ are the components of the metri-

of the metric tensor in the Lagrangian coordinate system (embedded in the medium) with basis vectors $\hat{e}_i$.

Using the results of works [8,9], one can find the structure of the nonlinear tensor relations (2) in the case where the arguments $\{A_{(r)}\}$, $\{B_{(s)}\}$ contain only scalars, vectors, and symmetric and antisymmetric tensors of rank three.

Thus, one can completely solve the problem of the structure of nonlinear dependences characterizing the physical properties of textures and crystals of the lower crystal systems.

It should be borne in mind that, from the standpoint of the general structure of tensor formulas, only the functionally independent invariants of the system of tensors $\{A_{(r)}\}$, $\{B_{(s)}\}$ and of the metric tensor are essential. It is also necessary to take into account that any tensor can be expressed as a linear combination of only p terms, where p is the number of its linearly independent components. In a number of published works, analogous formulas are considered in the form of sums in which the number of terms is substantially greater than the number p [8,9]. This is due to the fact that formulas containing only polynomial dependences were constructed.

For any tensor of rank n in three-dimensional Euclidean space, invariant with respect to a certain group G, the number $p \leqslant 3^n$ can be computed by the method of the theory of group representations (character theory) [3,4].

In what follows, formulas will be given that illustrate the structure of nonlinear relations for tensors characterizing the physical properties of textures and crystals.

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Note: Figure translations are in progress. See original paper for figures.