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Abstract

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SOME QUESTIONS ON THE OSCILLATION OF SOLUTIONS OF NONLINEAR NONAUTONOMOUS EQUATIONS OF THE SECOND ORDER*

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In the present article some questions of the theory of oscillation of solutions of nonlinear nonautonomous systems are considered from the point of view of the presence of many equilibrium positions. An attempt is made to classify the motions and to formulate some problems in the oscillation problem for the indicated systems. A class of equations is singled out possessing the property that the character of the oscillation of their solutions does not depend on the initial conditions (in works ($^{1-3}$), devoted to the study of oscillating (nonoscillating) solutions of nonlinear nonautonomous systems, these questions were not considered, since in ($^{1-3}$) either conditions are assumed which exclude the presence of many equilibrium positions, or systems with one equilibrium position are considered), and also some results are presented connected with the determination of domains of oscillation (nonoscillation) and the behavior of an analogue of the amplitude of oscillatory motions.

1. Consider the equation

$$(k(t)x')' + f(x, x', t) = 0 \tag{1}$$

(the prime denotes differentiation with respect to t) on the half-axis $[t_0, \infty)$ under the assumption of continuity of the functions $k(t) \geq k_0 > 0$ ($k_0 = \text{const}$), $k'(t)$, $f(x, x', t)$ with respect to their arguments and of the existence of a unique solution. We shall study the behavior of the solutions of equation (1) near equilibrium positions $x_r = \text{const}$ ($r = 1, 2, \ldots$), which are roots of the equation $f(x_r, 0, t) = 0$ for every t. We classify the solutions (motions) of system (1) according to the number of zeros of the function $y_r(t) = x(t) - x_r$, where a zero attained at infinity ($y_r(t) \to 0$ as $t \to \infty$) is excluded from consideration.

A solution of equation (1) will be called **nonoscillating** if $y_r(t)$, for every r, has no more than one zero. If $y_r(t)$ has one zero for each value of r, then the solution $x(t)$ will be called **rotational**. A solution $x(t)$ for which $y_r(t)$ ($r = 1, 2, \ldots, s$) has an infinite set of zeros will be called **oscillating**. If $y_r(t)$ has an infinite set of zeros only for the single $r = r_0$, then the solution $x(t)$ will be called **oscillating with respect to the given equilibrium position** x_{r_0}. An oscillating solution $x(t)$ with respect to a given equilibrium position x_{r_0} will be called **uniformly oscillating** if $y_r(t)$, for $r \neq r_0$, has not a single zero. A solution $x(t)$ for which $y_r(t)$ has n $(2 \leq n < \infty)$ zeros for a given $r = r_0$ will be called **mixed with respect to** x_{r_0}.

The set of initial values $G = G\{x_0, x_0'\}$ in which, for $t = t_0$, nonoscillating, rotational, oscillating, etc. solutions (motions) begin will be called the **domain of nonoscillating, rotational, oscillating**, etc. solutions (motions) of system (1).

We note that if system (1) admits a unique equilibrium position (as is the case for linear systems), then the notions of "oscillating," "oscillating with respect to a given position

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"equilibrium" and "uniformly oscillating" solutions coincide, and rotational motions do not occur. Let us emphasize that the forms of motion singled out here do not constitute a detailed classification of the motions characteristic of nonlinear nonautonomous systems. In particular, such interesting motions as periodic, almost periodic, and motions close to them have not yet been singled out.

On the basis of the definitions adopted, an analysis of the corresponding published works, and certain applied problems, it seems advisable to formulate the following main problems for equation (1):

1) To find conditions imposed on the functions $k(t)$ and $f(x, x', t)$ under which all solutions of equation (1) will be nonoscillating, rotational, oscillating, etc.
2) For fixed $k(t)$ and $f(x, x', t)$, to determine the domain of nonoscillating, rotational, oscillating, etc. solutions of equation (1).
3) For a given set of initial conditions, to establish a law of variation of the coefficients of equation (1) that guarantees a prescribed character of oscillation of its solutions.

Problem 1) was solved for various special cases of equation (1) in a number of works, in particular in ([1–3]). Problem 2) was solved for a nonlinear nonautonomous second-order equation close to a conservative one in works ([4,5]), and for systems close to Hamiltonian ones in work ([6]) (proceeding from the definition of oscillating solutions adopted in the indicated works).

2. Let us single out the class of equations

$$z'' + F(z, z', t) = 0, \qquad t \in [t_0, \infty), \tag{1'}$$

possessing the property that the values of the initial conditions do not affect the character of oscillation of their solutions, in the sense in which this follows from Sturm' s well-known theorem for linear nonautonomous second-order equations (we shall proceed here from the assumptions adopted for equation (1)).

Theorem 1. a) If $F(z,z',t)/z \geqslant \gamma_1 = 0$ ($\gamma_1 = \text{const}$) for all z, z', t, then the entire plane (z_0, z_0') is the domain of oscillating solutions of equation $(1')$; b) if $F(z,z',t)/z \leqslant -\gamma_2 < 0$ ($\gamma_2 = \text{const}$) for all z, z', t, then the entire plane (z_0, z_0') is the domain of nonoscillating solutions of equation $(1')$.

Remark 1. Assertion a) of Theorem 1 is also true in the case of equation (1), if $f(x,x',t)/x \geqslant \gamma > 0$ ($\gamma = \text{const}$) for all x, x', t and $0 < a \leqslant k(t) \leqslant b < \infty$ ($a = \text{const}, b = \text{const}$).

Remark 2. The conditions of Theorem 1 exclude the presence in system $(1')$ of many equilibrium positions.

Remark 3. Assertion b) of Theorem 1 is valid for equation $(1')$ under the condition $F(z,z',t)/z \leqslant 0$, and for equation (1), when $f(x,x',t)/x \leqslant 0$, $k(t) \geqslant a > 0$ (which, however, allow the presence of many equilibrium positions of systems (1), $(1')$).

Theorem 2. a) If $F(z,z',t)/z \geqslant (1+\alpha)/4t^2$ ($\alpha > 0$—const) for all z, z', t, then all solutions of equation $(1')$ are oscillating; b) if $F(z,z',t)/z \leqslant 1/4t^2$ for all z, z', t, then all solutions of equation (1) are nonoscillating.

Theorem 3. Suppose that in the equation

$$x'' + r(t)\varphi(x) = 0 \tag{1''}$$

$r(t) = O(1/t^{k+3})$, $k > 0$—const, $|\varphi(x_1) - \varphi(x_2)| \leqslant N|x_1 - x_2|$ ($N = \text{const}$) for any x_1, x_2. Then for any solution $x(t)$ of equation $(1'')$ the representation

$$x(t) - (\alpha t + \beta) = O\left(\frac{1}{t^k}\right) \qquad (\alpha, \beta\text{—const})$$

is valid.

Corollary. Fulfillment of the conditions of Theorem 3 guarantees the existence of an infinite set of solutions of equation $(1'')$ that are not oscillatory (in the sense adopted by us).

Theorem 3 is proved by means of the method of successive approximations, and moreover admits a generalization to the case $k > -1$.

Oscillation Theorems 1 and 2 are proved with the aid of the following comparison theorem for the equations

$$(k_1(t)u')' + f_1(u,u',t) = 0, \tag{2}$$

$$(k_2(t)v')' + f_2(v, v', t) = 0 \tag{3}$$

under the assumptions adopted for equation (1).

Theorem 4. *If $k_1 \geq k_2 > 0$, $f_2/v - f_1/u \geq 0$ for all u', v', $t \in [t_0, \infty)$, $u \in U(-\bar{u}, \bar{u})$, $v \in V(-\bar{v}, \bar{v})$, $0 < \bar{u} \leq \infty$, $0 < \bar{v} \leq \infty$, then between any two consecutive zeros of every solution $u \in U$ of equation (2) there lies at least one zero of any solution $v \in V$ of equation (3).*

This theorem is proved with the aid of Picone' s identity, obtained for equations (2), (3),

$$\left[\frac{u}{v}(k_1 u' v - k_2 u v')\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} u^2 \left(\frac{f_2}{v} - \frac{f_1}{u}\right) dt +$$

$$+ \int_{t_1}^{t_2} \left[(k_1 - k_2)u'^2 + k_2 \left(u' - \frac{u' v}{v}\right)^2\right] dt \tag{4}$$

and valid on any interval (t_1, t_2) on which $v(t) \neq 0$.

Assuming (without loss of generality) that the system ($1''$) admits the equilibrium position $x_r = 0$, as corollaries of Theorems 1, 2, and 4 we obtain the following assertions:

a) *If $\varphi(x)/x \geq \varphi_0 > 0$ ($\varphi_0 = \text{const}$) for $x \in X(-\bar{x}, \bar{x})$, $0 < \bar{x}$, $r(t) \geq r_0 > 0$ ($r_0 = \text{const}$), or if $\varphi(x)/x \geq \varphi_0 > 1/4$ ($\varphi_0 = \text{const}$) for $x \in X(-\bar{x}, \bar{x})$, $0 < \bar{x}$, $r(t) \geq (1+\alpha)/t^2$ ($\alpha > 0 -$ const), then the solutions $x(t) \in X$ of equation ($1''$) are uniformly oscillatory with respect to $x_r = 0$;* b) *if $\varphi(x)/x \leq 0$ for $x \in X(-\bar{x}, \bar{x})$, $0 < \bar{x}$, $r(t) > 0$, or if $\varphi(x)/x \leq 1/4$ for $x \in X(-\bar{x}, \bar{x})$, $0 < \bar{x}$, $0 \leq r(t) \leq 1/t^2$, then the solutions $x(t) \in X$ of equation ($1''$) will not be uniformly oscillatory with respect to $x_r = 0$.*

Using the preceding results, one establishes a theorem on the number of zeros of solutions of the nonlinear nonautonomous equation (3), if the number of zeros of solutions of the reference equation (2) is known.

Theorem 5. a) *Let $k_1 \geq k_2$, $f_2/v - f_1(u) \geq 0$ for arbitrary u, u', v, v', t; then, if the solution $u(t)$, $u_0 \neq 0$, $u'_0 \neq 0$, has m zeros on the interval $t_0 < t \leq t_0 + T$, there exist solutions $v(t)$ having on the same interval at least m zeros, and the k-th zero of $v(t)$ is less than the k-th zero of $u(t)$;* b) *let $k_1 \geq k_2$, $f_2/v - f_1/u \geq 0$ for $u \in U(-\bar{u}, \bar{u})$, $0 < \bar{u}$, $v \in V(-\bar{v}, \bar{v})$, $0 < \bar{v}$, for arbitrary u', v', t; then, if the solution $u(t) \in U$, $u_0 \neq 0$, $u'_0 \neq 0$, has m zeros on the interval $t_0 < t \leq t_0 + T$, then the solutions $v \in V$ have on the same interval at least m zeros, and the k-th zero of $v(t)$ is less than the k-th zero of $u(t)$.*

3. From the point of view of practical applications, it is of interest to establish conditions for the decay (growth) of an analogue of the amplitude of oscillatory motions. We formulate here two theorems for the equation

$$x'' + \alpha(t)x' + \beta(t)f(x) = 0 \tag{5}$$

under the assumption of the existence of expansions of the function $f(x)$ in a Taylor series in a neighborhood of $x = x_r$, with $f'(x_r) \neq 0$. By constructing a function of A. M. Lyapunov it is proved

Theorem 6. If $0 < \alpha_1 \leq \alpha(t) \leq \alpha_2 < \infty$, $0 < \delta_1 \leq \delta(t) \leq \delta_2 < \infty$ ($\delta(t) = \beta(t)f'(x_r)$), $4\alpha_1\delta_1(\alpha_1+\alpha_2) > (\delta_2-\delta_1)(\alpha_2^2+4\delta_1)$, then the equilibrium position $x = x_r$ is asymptotically stable.

Corollary. The fulfillment of the conditions of Theorem 6 guarantees the decrease (in any case, beginning with some τ, $t \geq \tau$) of the amplitude of the oscillatory motions of the system described by equation (5), and the amplitude will tend to zero as $t \to \infty$.

Analogously to (1), for equation (5), under the assumption that the function $F(x^2)$ is nondecreasing with respect to its argument

$$\left(F(x^2) = \int f(x)\,dx, \quad f(0) = 0\right),$$

one establishes

Theorem 7. If $F(x^2) > 0$ for $x \neq 0$, $|x| \leq c$, $c = \text{const}$, then the amplitude of solutions of equation (5) oscillating with respect to $x = 0$, $|x(t)| \leq c$, does not decay when $\beta' + 2\alpha\beta \leq 0$; it does not increase when $\beta' + 2\alpha\beta \geq 0$.

4. For the purpose of illustrating the corollaries of the formulated theorems, let us consider the motion of a nonautonomous pendulum

$$(ml^2\theta')' + mgl\sin\theta = 0 \tag{6}$$

under the assumption that its mass and length are bounded:

$$0 < m_1 \leq m(t) \leq m_2 < \infty, \qquad 0 < l_1 \leq l(t) \leq l_2 < \infty \quad (m_1, m_2, l_1, l_2 \text{ —const}).$$

In the linear approximation $\sin\theta \approx \theta$, as is known, all motions of the pendulum are uniformly oscillatory with respect to the single equilibrium position $\theta_0 = 0$. In the second approximation $\sin\theta \approx \theta - \theta^3/3!$, system (6) has three equilibrium positions: $\theta_0 = 0$, $\theta_{1,2} = \pm\sqrt{6}$. From the corollary of Theorem 2 it follows that, in this case, the solutions of equation (6) lying in the strip $(-\sqrt{6}, \sqrt{6})$ are uniformly oscillatory with respect to $\theta_0 = 0$. With respect to the equilibrium positions $\theta_{1,2} = \pm\sqrt{6}$, however, there are no oscillatory motions. It turns out that the next approximation $\sin\theta \approx \theta - \theta^3/3! + \theta^5/5!$ admits only one equilibrium position $\theta = 0$ and, consequently, by virtue of Theorem 1 all motions of the pendulum are uniformly oscillatory (thus, the third approximation is qualitatively worse in representing the dynamics of the pendulum). In the general case the equilibrium positions of system (6) are $\theta_r = r\pi (r = 0, \pm 1, \pm 2, ...)$. On the basis of the corollary of Theorem 2 we arrive at the conclusion that, if solutions of equation (6) exist in the strips $((2r-1)\pi, (2r+1)\pi)$, then they will be uniformly oscillatory with respect to $\theta_r = 2r\pi$. With respect to the equilibrium positions $\theta_r = (2r+1)\pi$ there can be no oscillatory motions.

It follows from Theorems 1, 4, and 7 that the uniformly oscillatory solutions of equation (6), in the case of all the approximations considered, do not decay when the function $s(t) = m^2 l^3$ does not increase, and do not increase when $s'(t)$ does not decrease.

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Note: Figure translations are in progress. See original paper for figures.