DIFFERENTIABILITY OF THE NEMYTSKII OPERATOR

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Abstract

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DIFFERENTIABILITY OF THE NEMYTSKII OPERATOR

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It is well known what an important role the theory of branching points plays in nonlinear analysis. Important results in this direction have been obtained by M. A. Krasnosel' skii and others. In applying this theory to nonlinear integral operators of Hammerstein type, the question arises of conditions for differentiability of the Nemytskii operator at zero in a certain Banach space. In the present note we study the question of differentiability of this operator in Orlicz spaces.

1. Let G be an L-measurable bounded set in a finite-dimensional Euclidean space. Let the function $f(t,u)$ ($t \in G$, $-\infty < u < \infty$) satisfy the Carathéodory conditions: it is measurable in t for each u and continuous in u for almost all t. The operator $f\varphi(t) = f[t, \varphi(t)]$ is called the Nemytskii operator. In the present note we shall assume, without loss of generality, that $f(t,0) \equiv 0$. (For all definitions and assertions from the theory of Orlicz spaces used below, see ([1]).)

Let the N-function $M(u)$ grow essentially faster than the N-function $M_1(u)$. Then, as Andô showed ([2]), there exists an N-function $\widetilde{M}(u)$ such that the product $\varphi(t)\psi(t)$ belongs to the space $L^*_{M_1}$ for any pair of functions $\varphi(t) \in L^*_M$, $\psi(t) \in L^*_{\widetilde{M}}$. This N-function $\widetilde{M}(u)$ can be defined by the equality

$$\widetilde{M}(u) = \sup\{M_1(uv) - M(v) : v \geqslant 0\}. \tag{1}$$

Lemma 1. Let $M(u)$ grow essentially faster than $M_1(u)$. Then for any $C \geqslant 1$ the N-function

$$\widetilde{M}_C(u) = \sup\{M_1(uv) - CM(v) : v \geqslant 0\}$$

is equivalent to $\widetilde{M}(u)$.

Lemma 2. Let the N-function $M(u)$ grow essentially faster than the N-function $M_1(u)$. Let $g(t)$ be some measurable function. In order that $g(t)\varphi(t)$ belong to the space $L^*_{M_1}$ for every function $\varphi(t)$ from L^*_M, it is necessary and sufficient that the function $g(t)$ belong to the space $L^*_{\widetilde{M}}$, where the N-function $\widetilde{M}(u)$ is defined by equality (1). Moreover,

$$\|g(t)\varphi(t)\|_{M_1} \leqslant a\,\|g(t)\|_{\widetilde{M}}\|\varphi(t)\|_M.$$

2. In this section we consider necessary conditions for differentiability at zero of the operator f.

Lemma 3. Let the operator f act from L_M^* into $L_{M_1}^*$ and be differentiable at zero. Then there exists a measurable function $g(t)$ satisfying the condition

$$\int_G M_1[F_C(\lambda|g(t)|)]\;dt < \infty, \tag{2}$$

that

$$\mathfrak{G}h(t) = g(t)h(t) \tag{3}$$

is the differential of the operator f at zero.

Here $F_c(v)$ denotes the function $F_c(v) = vf_c(v)$, where $f_c(v) = \sup\{u \geq 0 : M_1(uv) \geq cM(u)\}$. The function $F_c(v)$ was introduced by I. V. Shragin [(3)], who studied conditions for continuity of the operator $\mathfrak{G}$ defined by equality (3).

It is easy to see that $f(t,u)/u$ tends in measure to $g(t)$ as $u \to 0$. Let $G_0 \subset G$ be the set of those points t for which the equality

$$f(t,u) = g(t)u \tag{4}$$

does not hold at least for some u.

Lemma 4. If $\operatorname{mes} G_0 > 0$, then the N-function $M(u)$ grows essentially faster than the N-function $M_1(u)$.

Corollary 1. If $M(u)$ does not grow essentially faster than $M_1(u)$, then equality (4) holds for almost all $t \in G$, where $g(t)$ satisfies condition (2).

Corollary 2. If $M(u) \sim M_1(u)$, then the function $g(t)$ is bounded; if $M(u) < M_1(u)$ and $M(u)$ is not equivalent to $M_1(u)$, then $f(t,u) = 0$ almost everywhere.

From the formulated lemmas it follows that

Theorem 1. Let the operator f act from L_M^* into $L_{M_1}^*$. If f is differentiable at zero, then the following alternative holds: either the operator f is linear, or the N-function $M(u)$ grows essentially faster than the N-function $M_1(u)$. In the latter case $f(t,u)/u$ tends in measure to the function $g(t)$ as $u \to 0$, and $g(t) \in L_{\widetilde{M}}^*$, where $\widetilde{M}(u)$ is defined by equality (1).

3. Let us consider some sufficient conditions for differentiability of the operator f. Obviously, it suffices to consider the case when the N-function $M(u)$ grows essentially faster than the N-function $M_1(u)$. Put

$$\begin{aligned} &\eta(t,u) = f(t,u)/u - g(t), \quad \text{when } u \neq 0,\ t \in G; \\ &\eta(t,u) = 0, \qquad\qquad\qquad\quad \text{when } u = 0,\ t \in G. \end{aligned} \tag{5}$$

If the operator f is differentiable at zero, then the function $\eta(t,u)$ satisfies the following generalized Carathéodory condition: $\eta(t,u)$ is measurable in t for each u and is continuous in u for $u \neq 0$ for almost all $t \in G$, and $\eta(t,u)$ tends to $\eta(t,0)$ in measure as $u \to 0$.

Denote by $T(\theta,\gamma,L_M^*)$ the ball $\|u\|_M < \gamma$ of the space L_M^*.

Theorem 2. Let the function $\eta(t,u)$ satisfy the generalized Carathéodory condition. Let $g(t) \in L^*_{\widetilde{M}}$. Let the operator $\eta\varphi(t) = \eta(t,\varphi(t))$ act from $T(\theta,\gamma,L_M^*)$ into L_M^* and be continuous at zero. Then the operator f acts from $T(\theta,\gamma,L_M^*)$ into $L^*_{M_1}$ and is differentiable at zero. The differential of this operator is determined by formula (3).

Corollary 1. Suppose that for all $t \in G$, $-\infty < u < \infty$, there exists $f'_u(t,u)$. Suppose the functions $f(t,u)$, $f'_u(t,u)$ satisfy the Carathéodory conditions. Suppose the operator $f_1\varphi(t) = f'_u[t,\varphi(t)]$ acts from $T(\theta,\gamma,L_M^*)$ into $L^*_{\widetilde{M}}$ and is continuous at zero. Then the operator f acts from $T(\theta,\gamma,L_M^*)$ into $L^*_{M_1}$ and is differentiable at zero.

Corollary 2. Let the function $\eta(t,u)$ satisfy the generalized Carathéodory condition. Let $1/p_1 + 1/p_1' = 1/p_2$. If the operator $\eta\varphi(t) = \eta(t,\varphi(t))$ acts from L^{p_1} into $L^{p_1'}$, then the operator f acts from L^{p_1} into L^{p_2} and is differentiable at zero.

Theorem 3. Let $f(t,u)$, $f'_u(t,u)$ satisfy the Carathéodory conditions. Let $f(t,u)$ be nonlinear in u. In order that the operator f act-

act from $T(\theta,\gamma,L_M^*)$ into $L^*_{M_1}$ and be continuously differentiable at every point of $T(\theta,\gamma,L_M^*)$, it is necessary and sufficient that the following conditions hold: 1) the N-function $M(u)$ grows essentially faster than the N-function $M_1(u)$; 2) the operator $f_1\varphi(t) = f'_u[t,\varphi(t)]$ acts from $T(\theta,\gamma,L_M^*)$ into $L^*_{\widetilde{M}}$ and is continuous at every point of the ball $T(\theta,\gamma,L_M^*)$.

Corollary. Let $f(t,u)$, $f'_u(t,u)$ satisfy the Carathéodory conditions. Let $f(t,u)$ be nonlinear in u. In order that the operator f acting from L^{p_1} into L^{p_2} be differentiable at every point of the space L^{p_1}, it is necessary and sufficient that the following conditions hold: 1) $p_1 > p_2$; 2) the operator f_1 acts from L^{p_1} into $L^{p_1'}$, where $1/p_1 + 1/p_1' = 1/p_2$.

From the assertions proved there follow the theorem of M. A. Krasnosel' skii and Ya. B. Rutitskii ([1]) on conditions for differentiability of the operator f acting in Orlicz spaces, and the theorem of M. M. Vainberg ([4]) on conditions for differentiability of the operator f acting in spaces L^p.

4. The differentiability conditions for the operator f make it possible to indicate conditions for differentiability of the Hammerstein operator. We

shall consider the more general operator

$$Kf\varphi(s) = \int_G K(s,t)\, f[t, \varphi_1(t), \dots, \varphi_n(t)]\, dt, \tag{6}$$

where $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))$ is a vector-function.

Theorem 4. Let $M(u), M_1(u), \Phi(u)$ be N-functions, and $\Psi(v)$ an N-function complementary to $\Phi(u)$. Suppose that for some $\alpha, u_0 > 0$

$$\Phi(\alpha uv) \le M_1(u) N(v) \quad (u, v \ge u_0),$$

where $N(v)$ is an N-function complementary to $M(u)$. Suppose the following conditions are satisfied: 1) $K(t,s) \in E_\Psi(G \times G)$; 2) there exist n bounded measurable functions $g_1(t), \dots, g_n(t)$ such that

$$[f(t, u_1, \dots, u_i, \dots, u_n) - f(t, u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_n)]/u_i$$

tends in measure to $g_i(t)$ as $u_i \to 0$ and $|u_j| \le |u_i|$ $(j \ne i)$; 3) the operator f acts from $\mathcal{E}_M = E_M + \dots + E_M$ into E_{M_1} and there exists a constant b such that

$$|f(t, u_1, \dots, u_n) - f(t, 0, \dots, 0)| \le b \sum_{i=1}^n |u_i|.$$

Then the operator (6) is differentiable at zero, and its differential has the form

$$Bh(s) = \sum_{i=1}^n K\mathcal{G}_i h(s) = \int_G K(s,t) \sum_{i=1}^n g_i(t) h_i(t)\, dt.$$

If the positive definite kernel $K(s,t)$ belongs to $L^2(G \times G)$ and the operator f is defined in L^2 and satisfies the hypotheses of Theorem 4, then it is easy to prove the differentiability of the operator HfH, where H is the positive square root of the linear operator K. This fact finds applications in the theory of bifurcation points developed by M. A. Krasnosel' skii ([5]).

5. Let us now consider another sufficient condition for differentiability of the operator f. Let a function $R(u) \ge 0$ be defined on $[0, \infty)$. We shall say that $R(u)$ satisfies condition (Δ) if, for every $\varepsilon > 0$, one can specify $\delta, u_0 > 0$ such that

$$R(u) \le \varepsilon\gamma R(u/\gamma)$$

for all $u \ge u_0,\ 0 \le \gamma \le \delta$.

Lemma 5. Let $R(u)$ be a nondecreasing, nonnegative, differentiable function, $R'(u) = \gamma(u)$. Suppose that for every $N > 0$ there exist $l \ge 1,\ u_0 > 0$ such that

$$\gamma(\eta u)/\gamma(u) \ge N$$

for $\eta \geq l,\ u \geq u_0$. Then $R(u)$ satisfies condition (Δ).

Examples of functions satisfying condition (Δ) are

$$R_1(u) = u^\gamma \quad (\gamma > 1,\ u \geq 0), \qquad R_2(u) = u^\gamma(|\ln u| + 1) \quad (\gamma > 1,\ u > 0),$$

$$R_3(u) = e^u - u - 1 \quad (u \geq 0), \qquad R_4(u) = e^{u^\beta} - 1 \quad (\beta > 1,\ u \geq 0).$$

It is easy to see that a nondecreasing positive function $R(u)$, satis-

satisfying condition (Δ), grows faster than some power function $u^{1+\alpha}$ $(\alpha > 0)$. An example of a nondecreasing function not satisfying condition (Δ) is the function $R(u) = u|\ln u|$ $(u > 0)$.

Theorem 5. *Let the following conditions be satisfied: 1) $f(t,u)/u$ tends to $g(t)$ uniformly in $t \in G$ as $u \to 0$, where $g(t) \in L^*_{\widetilde{M}}$; 2) there exists a function $R(u) \geq 0$ $(0 \leq u < \infty)$, satisfying condition (Δ), and such that*

$$R(|u|) \leq bM_1^{-1}[M(ku)] \qquad (-\infty < u < \infty), \tag{7}$$

$$|f(t,u) - g(t)u| \leq a + R|u| \qquad (t \in G,\ -\infty < u < \infty),$$

where a, b, k are some positive constants.

Then the operator f maps some neighborhood of zero of the space L^*_M into $L^*_{M_1}$ and is differentiable at zero. Its differential is determined by formula (3).

Corollary. *Let $1/p_1 + 1/p_1' = 1/p_2$ $(p_2 > 1)$. Let*

$$|f(t,u) - g(t)u| \leq a + b|u|^{p_1/p_2} \qquad (t \in G,\ -\infty < u < \infty),$$

where $g(t) \in L_{p_1'}$, and a, b are constants. Then the operator f maps L^{p_1} into L^{p_2} and is differentiable at zero.

A consequence of Theorem 5 is also the theorem of M. A. Krasnosel' skii and Ya. B. Rutitskii [1] on conditions for differentiability of the operator f at a single point of an Orlicz space.

6. The use of Theorem 5 from [6] makes it possible to prove the following assertion:

Theorem 6. *Let $f(t,u)$, $f_u'(t,u)$ satisfy the Carathéodory conditions. Let the N-function $M(u)$ grow essentially faster than the N-function $M_1(u)$. If the operator f is continuously differentiable in the ball $T(0, r, L^*_M)$, then it is also continuously differentiable in $\Pi(r, E_M)$.*

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Note: Figure translations are in progress. See original paper for figures.