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Abstract

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ON THE STRUCTURE OF THE RING OF FINITE FUNCTIONS ON THE GROUP OF UNIMODULAR MATRICES OF SECOND ORDER WITH ELEMENTS FROM A TOTALLY DISCONNECTED LOCALLY COMPACT FIELD

1. We consider the group G of unimodular matrices of order 2 with elements from a totally disconnected locally compact nondiscrete field K. On the group G consider the totality Γ of functions $f(g)$, $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, satisfying the following conditions:

a) **Finiteness condition.** There exists a number $N > 0$ (depending on f) such that $f(g) = 0$ whenever

$$\|g\| \equiv \max(|\alpha|, |\beta|, |\gamma|, |\delta|) > N.$$

(Here $|\alpha|$ denotes the norm of the element α.)

b) For every element g of G one can specify a neighborhood U_g of it in which the function $f(g)$ is constant.*

We note that for finite functions $f(g)$ condition b) is equivalent to the following condition:

b′) There exists an open subgroup G_0 of the group G (depending on f) such that

$$f(g_1 g g_2) = f(g)$$

for every g in G and arbitrary g_1, g_2 in the subgroup G_0.

If multiplication in Γ is defined as convolution and the topology is introduced in a natural way, then we obtain a ring. The purpose of the present paper is to study the structure of the ring Γ.**

2. First we recall the description of the continuous and discrete series of representations of the group G, obtained in ([3]). A representation of the continuous series is specified by a multiplicative character $\pi(x)$ on K. It is constructed in the space $\mathfrak{D}_\pi$ of functions $\varphi(x)$ on K satisfying the following condition: there exists an open subgroup G_0 of the group G such that, for every

$$g_0 = \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix}$$

from G_0, we have

$$\pi(\gamma_0 x + \alpha_0)\, |\gamma_0 x + \alpha_0|^{-1} \varphi\left(\frac{\delta_0 x + \beta_0}{\gamma_0 x + \alpha_0}\right) \equiv \varphi(x). \tag{1}$$

The representation operator $T_\pi(g)$, corresponding to the matrix

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

is given by the formula

$$T_\pi(g)\varphi(x) = \pi(\gamma x + \alpha)\, |\gamma x + \alpha|^{-1} \varphi\left(\frac{\delta x + \beta}{\gamma x + \alpha}\right)^{***}. \tag{2}$$

For the description of the representations of the discrete series we shall need several definitions and notations (see ([3])). Let O be the ring of integral elements –

* This condition is the natural replacement, in the case of zero-dimensional groups, for the condition of infinite differentiability.

** For connected fields an analogous investigation was carried out earlier in ([4,7]) for the case of the field of complex numbers and in ([8]) for the case of the field of real numbers. We also note the work ([5]), in which the structure of the ring of infinitely differentiable rapidly decreasing functions on an arbitrary complex semisimple Lie group is studied.

*** The space $\mathfrak{D}_\pi$ can also be realized as the space of functions $f(x_1, x_2)$ satisfying the homogeneity condition

$$f(tx_1, tx_2) = \pi(t)|t|^{-1} f(x_1, x_2).$$

Condition (1) is then replaced by the condition

$$f(\alpha_0,\ x_1 + \gamma_0 x_2,\ \beta_0 x_1 + \delta_0 x_2) \equiv f(x_1, x_2).$$

The operator $T_\pi(g)$ is given by the formula

$$T_\pi(g) f(x_1, x_2) = f(\alpha x_1 + \gamma x_2,\ \beta x_1 + \delta x_2).$$

The passage from one realization of the space $\mathfrak{D}_\pi$ to the other is effected by the relation $\varphi(x) = f(1, x)$.

elements in K, P is the maximal ideal in O, q is the order of the residue field O/P. Denote by $\mathfrak{p}$ a generator of the ideal P, and by ε an element of the field K of order $q-1$ ($\varepsilon^{q-1}=1$).

We shall assume that the characteristic of the field K is not equal to two*. Under this assumption the field K has exactly 3 quadratic extensions, namely $K(\sqrt{\mathfrak{p}})$, $K(\sqrt{\varepsilon\mathfrak{p}})$, and $K(\sqrt{\varepsilon})$. Consider, in the field $K(\sqrt{\tau})$, $\tau=\mathfrak{p},\varepsilon\mathfrak{p},\varepsilon$, the set C_τ of elements $t=x+\sqrt{\tau}\,y$ such that $t\bar{t}=x^2-\tau y^2=1$. The set C_τ is a group under multiplication; denote by $\pi_\tau(t)$ the characters of the group C_τ. From the compactness of C_τ it follows that the set of characters is discrete. Each character $\pi_\tau(t)$ we extend in some way to a multiplicative character on $K(\sqrt{\tau})$; this character on $K(\sqrt{\tau})$ will also be denoted by $\pi_\tau(t)$.

With each quadratic extension $K(\sqrt{\tau})$ of the field K there is associated a discrete series of representations of the group G. A representation of this series is determined by a character $\pi_\tau(t)\neq 1$ on C_τ**. It is constructed in the space H of functions $\varphi(u)$ on K for which $\int|\varphi(u)|^2\,du<+\infty$. The representation operator $T_{\pi_\tau}(g)$, corresponding to the matrix $g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, is given by the formula

$$T_{\pi_\tau}(g)\varphi(u)=\int K(g\mid u,v\mid\pi_\tau)\varphi(v)\,dv, \tag{3}$$

where

$$K(g\mid u,v\mid\pi_\tau)=c_\chi\frac{\operatorname{sign}_\tau\gamma}{|\gamma|}\operatorname{sign}_\tau u\cdot\chi\Big(\frac{\alpha u+\delta v}{\gamma}\Big)\times$$

$$\times\int_{\bar{t}t=vu^{-1}}\chi\Big(-\frac{1}{\gamma}(ut+vt^{-1})\Big)\,\pi_\tau(t)\,d^*t, \tag{4}$$

if $\gamma\neq 0$, $\operatorname{sign}_\tau u=\operatorname{sign}_\tau v$; $K(g\mid u,v\mid\pi_\tau)=0$, if $\gamma\neq 0$, $\operatorname{sign}_\tau u\neq\operatorname{sign}_\tau v$; $K(g\mid u,v\mid\pi_\tau)=\operatorname{sign}_\tau\alpha\cdot\pi(\alpha)|\alpha|\chi(\alpha\beta u)\delta(\alpha^2u-v)$, if $\gamma=0$. Here $\operatorname{sign}_\tau x$ is a multiplicative character on K: $\operatorname{sign}_\tau x=1$, if $x=t\bar{t}$, $t\in K(\sqrt{\tau})$, and $\operatorname{sign}_\tau x=-1$, if x is not representable in the form $t\bar{t}$. By $\chi(x)$ is denoted a fixed additive character on K, $\chi(x)\neq 1$, and c_χ is a numerical factor defined in (3).

3. To each function $f(g)\in\Gamma$ assign the operators

$$T_\pi(f)=\int f(g)T_\pi(g)\,dg$$

and

$$T_{\pi_\tau}(f)=\int f(g)T_{\pi_\tau}(g)\,dg,\qquad \tau=\mathfrak{p},\varepsilon\mathfrak{p},\varepsilon.$$

Denote by $K_1(x,y\mid\pi)$ the kernel of the operator $T_\pi(f)$, and by $K_\tau(u,v\mid\pi_\tau)$ the kernel of the operator $T_{\pi_\tau}(f)$. The collection of kernels $K_1,K_{\mathfrak{p}},K_{\varepsilon\mathfrak{p}},K_\varepsilon$ will be called the Fourier transform of the function $f(g)$.

Theorem. The mapping

$$f(g) \to \{K_1, K_{\mathfrak{p}}, K_{\varepsilon\mathfrak{p}}, K_\varepsilon\}$$

realizes an isomorphism of the ring Γ onto the direct sum $\Gamma_1 \oplus \Gamma_{\mathfrak{p}} \oplus \Gamma_{\varepsilon\mathfrak{p}} \oplus \Gamma_\varepsilon$ of four rings of kernels. Below a description of these rings is given.

The ring Γ_1 consists of kernels $K(x, y \mid \pi)$ satisfying the following conditions:

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* Thus, we do not consider the case where K is a field of characteristic 2, nor the case where K contains the subfield of 2-adic numbers.

** In the cases $\tau = \mathfrak{p}$ and $\tau = \varepsilon\mathfrak{p}$ it is additionally assumed that $\pi_\tau(t) \neq 1$ on the subgroup of index 2 of the group C_τ, determined by the condition $|1 - t| < 1$.

a) There exists an open subgroup G_0 of the group G such that, for any matrices $g_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}$ from G_0, we have

$$\pi(\gamma_1 x+\alpha_1)\, |\gamma_1 x+\alpha_1|^{-1} \pi^{-1}(\gamma_2 y+\alpha_2)\, |\gamma_2 y+\alpha_2|^{-1} \times K\left(\frac{\delta_1 x + \beta_1}{\gamma_1 x + \alpha_1}, \frac{\delta_2 y + \beta_2}{\gamma_2 y + \alpha_2} \,\middle|\, \pi\right) = K(x, y \mid \pi). \tag{5}$$

In other words, as a function of x and y the kernel $K(x, y \mid \pi)$ belongs to the tensor product $\mathcal{D}_\pi \otimes \mathcal{D}_{\pi^{-1}}$ of the spaces $\mathcal{D}_\pi$ and $\mathcal{D}_{\pi^{-1}}$.

b) There exists a function $\varphi(x, y; t)$ such that

$$K(x, y \mid \pi) = \int \varphi(x, y; t)\, \pi(t)\, |t|^{-1}\, dt. \tag{6}$$

This function satisfies the following conditions. There exist positive constants c_1, c_2 ($c_1 < c_2$) such that $\varphi(x, y; t) = 0$ when $|t| \max(1, |y|) < c_1 \max(1, |x|)$ and when $|t| \max(1, |y|) > c_2 \max(1, |x|)$. Furthermore, the estimate

$$|\varphi(x, y; t)| = O\left([\max(1, |x|) \cdot \max(1, |y|)]^{-1}\right)$$

holds.

c) The kernels $K(x, y \mid \pi)$ and $K(x, y \mid \pi^{-1})$ are connected by a relation. Namely, there exists an operator B_π such that $B_\pi K(x, y \mid \pi) = K(x, y \mid \pi^{-1}) B_\pi$. The concrete form of this relation is the following:

$$\int K(x - t, y \mid \pi)\, \pi^{-1}(t)\, |t|^{-1}\, dt = \int K(x, y - t \mid \pi^{-1})\, \pi^{-1}(t)\, |t|^{-1}\, dt. \tag{7}$$

The integrals should be understood in the sense of the regularized value, cf. (3). Multiplication in Γ_1 is defined as the ordinary convolution of kernels:

$$K' * K'' = \int K'(x, t \mid \pi)\, K''(t, y \mid \pi)\, dt.$$

The ring Γ_τ, $\tau = \rho, \varepsilon\rho, \varepsilon$, consists of kernels $K(u, v \mid \pi_\tau)$ satisfying the following conditions*:

a) The kernel $K(u, v \mid \pi_\tau)$ is a finite continuous function of u and v, equal to zero when $\operatorname{sign}_\tau u \neq \operatorname{sign}_\tau v$.

b) There exists an open subgroup G_0 of the group G such that, for any matrices g_1 and g_2 from G_0, we have

$$\int K(g_1 \mid u, u' \mid \pi_\tau)\, K(g_2 \mid v, v' \mid \pi_\tau^{-1})\, K(u', v' \mid \pi_\tau)\, du'\, dv' \equiv K(u, v \mid \pi_\tau). \quad (8)$$

c) $K(u, v \mid \pi_\tau) = 0$ for all characters π_τ, except for a finite number of them.

d) The kernels $K(u, v \mid \pi_\tau)$ and $K(u, v \mid \pi_\tau^{-1})$ are connected by the relation

$$\pi_\tau(u)\, K(u, v \mid \pi_\tau) = \pi_\tau(v)\, K(u, v \mid \pi_\tau^{-1}). \quad (9)$$

Multiplication in Γ_τ is defined as the ordinary convolution of kernels:

$$K' * K'' = \int K(u, w \mid \pi_\tau)\, K(w, v \mid \pi_\tau)\, dw.$$

The listed conditions on the kernels are easily derived from formulas expressing these kernels through the function $f(g)$. The further proof of the theorem is based on the inversion formula and the Plancherel formula for functions on G, obtained in (2) (see also (3)).

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* Let us note that conditions a) and b) are dependent. Namely, from condition b), under the assumption that integral (8) converges, it follows that the function $K(u, v \mid \pi_\tau)$ is finite.

We note that the theorem formulated above differs essentially from the analogous theorem for the case of a connected field K (i.e., for the field of real or the field of complex numbers). Namely, in the case of a connected field K, the ring Γ of finite infinitely differentiable functions on G is isomorphic to the ring Γ_1 of germs $K(x, y \mid \pi)$. But the germs $K(x, y \mid \pi)$ themselves, in the case of a connected field K, have a more complicated structure: in addition to conditions analogous to those listed in the present paper, they satisfy further relations arising at the "integral" points π.

4. Using the results of the present article, one can prove the following theorem.

Theorem. *The unitary representations of the group G described in* $(1, 3)$ *exhaust all irreducible unitary representations of this group.*

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Note: Figure translations are in progress. See original paper for figures.