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Abstract

Full Text

MATHEMATICS

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On the Growth of Functions Harmonic in a Cylinder and Growing on Its Boundary Together with the Normal Derivative

(Presented by Academician M. V. Keldysh on 20 IV 1963)

In the papers of M. A. Evgrafov and I. S. Arshon ^(1,2), for harmonic functions of three variables a theorem was proved analogous to the Phragmén–Lindelöf theorem for analytic functions, which may be formulated as follows.

Theorem. Let $U(x, x_1, x_2)$ be a harmonic function of three variables in the half-cylinder

$$0 < x < \infty, \quad (x_1, x_2) \subset D, \quad (1)$$

continuous together with its first-order partial derivatives up to its surface.

Suppose that the boundary of the two-dimensional domain D satisfies the Lyapunov conditions. Denote by σ the width of the smallest layer between planes parallel to the x -axis in which the given cylinder can be placed.

If on the lateral surface of the half-cylinder (1) the function $H(x, x_1, x_2)$ satisfies the conditions

$$|U(x, x_1, x_2)| + \left| \frac{\partial U(x, x_1, x_2)}{\partial n} \right| = O(x^{-\alpha}), \quad 0 < \alpha < 1, \quad x \rightarrow \infty, \quad (2)$$

and inside it the condition

$$|U(x, x_1, x_2)| < M \exp \exp \frac{\pi}{\sigma + \eta} x, \quad \eta > 0, \quad x > 0, \quad (3)$$

then

$$U(x, x_1, x_2) = \frac{1}{4\pi} \iint_{\Gamma} \left[\frac{1}{r} \frac{\partial U}{\partial n} - U \frac{\partial (\frac{1}{r})}{\partial n} \right] ds, \quad (4)$$

$$|U(x, x_1, x_2)| = O(x^{-\alpha}), \quad x \rightarrow \infty \quad (5)$$

uniformly with respect to $(x_0, x_2) \subset D$.

Here Γ is the boundary of the half-cylinder (1), ds is the area element, and r is the distance from any interior point (x, x_1, x_2) of the half-cylinder to its boundary Γ .

In the work ⁽³⁾ of the same authors, this result was extended to a broader class of domains. M. A. Evgrafov posed the following problems:

1. To prove an analogous theorem for the case when the harmonic function $U(x, x_1, x_2)$ on the lateral surface of the half-cylinder (1) grows together with its normal derivative.
2. To weaken condition (3).

When the domain D is a circle or a rectangle, the first problem was partially solved by A. F. Leont'ev, and the second was partially solved by I. A. Chegis in her dissertation. In the present note a complete solution of the posed problems is given.

Without loss of generality, we may assume that the domain D contains the point $(0, 0, 0)$ and is contained inside the layer determined by the inequa-

$$-\frac{\pi}{2\rho_1} \leq x_1 \leq \frac{\pi}{2\rho_1}, \quad 0 < \rho_1 < \infty.$$

Theorem 1. If a function $U(x, x_1, x_2)$, harmonic in the half-cylinder, satisfies inside the half-cylinder the condition

$$|U(x, x_1, x_2)| < M \exp(c \exp \rho_1 x), \quad 0 < c < \beta, \quad x > 0, \quad (6)$$

then

$$U(x, x_1, x_2) = \frac{1}{4\pi} \iint_{\Gamma} \left[\frac{\partial(\Phi_1 + \Phi_2)}{\partial n} U - (\Phi_1 + \Phi_2) \frac{\partial U}{\partial n} \right] ds, \quad \rho < \rho_1, \quad (7)$$

where Φ_1 and Φ_2 are defined by the equalities

$$\begin{aligned} \Phi_1 = & \frac{1}{(2\pi i)^2} \int_{\pi/2\rho_1, x-ix_1} \int_{L_{x+|x-t|-it_1}} \int_0^\infty \varphi(s - it_1, x + |x - t|) \cos(t_2 - x_2) \xi_2 \\ & \times \psi(\tau - ix_1) \exp(-\alpha e^{-\rho s} - \beta e^{-\rho_1 s} + \alpha e^{-\rho \tau} + \beta e^{-\rho_1 \tau}) \frac{d\xi_2 d\tau ds}{\tau - s}, \end{aligned}$$

$$\Phi_2 = \frac{1}{(2\pi i)^2} \int_{\pi/2\rho_1, x+ix_1} \int_{L_{x+|x-t|+it_1}} \int_0^\infty \varphi(s - it_1, x + |x - t|) \cos(t_2 + x_2) \xi_2$$

Fig. 1

Figure 1: Fig. 1

$$\times \psi(\tau + ix_1) \exp(-\alpha e^{-\rho s} - \beta e^{-\rho_1 s} + \alpha e^{-\rho \tau} + \beta e^{-\rho_1 \tau}) \frac{d\xi_2 d\tau ds}{\tau - s},$$

Fig. 1

$$\varphi(s, \xi) = \int_0^\infty \frac{\exp[-\xi \sqrt{u^2 + \xi_2^2}]}{\sqrt{u^2 + \xi_2^2}} \exp[-su] du,$$

$$\psi(\tau) = \int_0^\infty \exp[\tau \xi_1 + x \sqrt{\xi_1^2 + \xi_2^2}] d\xi_1.$$

The contour L_{x+iy} ($x > 0$, $|y| < \pi/2\rho_1$) comes from infinity along the lower bank of the cut $(-\infty - iy, -x - iy)$ to the point $-x - \varepsilon - iy$, goes around the point $-x - iy$ along a circle of radius $\varepsilon > 0$, and along the upper bank of the cut $(-\infty - iy, -x - iy)$ again goes to infinity.

The contour $l_{\pi/2\rho_1, x+iy}$ comes from infinity along the half-line

$$\left(-\infty - \frac{\pi}{2\rho_1}i, -\frac{\pi}{2\rho_1}i\right]$$

to the point $-\frac{\pi}{2\rho_1}i$, then along the imaginary axis to the point $-iy - i\varepsilon$, then along the line parallel to the real axis to the point $-x - i(y + \varepsilon)$, goes around the point $-x - iy$ along a circle of radius $\varepsilon > 0$ to the point $-x - i(y - \varepsilon)$, after which it again goes parallel to the real axis to its intersection with the imaginary axis at the point $-iy + i\varepsilon$, then along the imaginary axis goes to the point $\frac{\pi}{2\rho_1}i$ and, finally, along the half-line

$$\left(-\infty + \frac{\pi}{2\rho_1}i, \frac{\pi}{2\rho_1}i\right]$$

again goes to infinity (see Fig. 1).

The theorem is proved by a slight modification of the method proposed in paper (1).

Putting $\alpha = 0$ in formula (7), and then letting β tend to zero, we obtain the solution of the second problem.

Theorem 2. If a function $U(x, x_1, x_2)$, harmonic in the half-cylinder (1), satisfies on its lateral surface the conditions (2), and inside the half-cylinder the condition

$$|U(x, x_1, x_2)| < M \exp \left\{ o \left(\exp \frac{\pi}{\rho} \right) \right\}, \quad x > 0,$$

then for the function $U(x, x_1, x_2)$ formula (4) is valid, and inside the semicylinder estimate (5) holds.

Further, letting β tend to zero in formula (7), we obtain the solution of the first problem.

Theorem 3. If a function harmonic in the semicylinder (1) satisfies on the boundary of this semicylinder the conditions

$$|U(x, x_1, x_2)| < M \exp(a(t_1, t_2) \exp \rho x), \quad x > 0, \quad \rho < \rho_1,$$

$$\left| \frac{\partial U(x, x_1, x_2)}{\partial n} \right| < M \exp(a(t_1, t_2) \exp \rho x), \quad x > 0, \quad \rho < \rho_1,$$

where $a(t_1, t_2)$ is a positive function given on the boundary of the domain D , and inside the semicylinder the condition

$$|U(x, x_1, x_2)| < M \exp\{o(\exp \rho_1 x)\}, \quad x > 0,$$

holds, then

$$|U(x, x_1, x_2)| < M \exp(a_1 \cos \rho x_1 \exp \rho x), \quad x > 0, \quad (8)$$

uniformly with respect to $(x_1, x_2) \in D$, where

$$a_1 = \max_{t_1} \frac{a(t_1, t_2)}{\cos \rho t_1}.$$

The example of the harmonic function

$$u = \operatorname{Re} [\exp b \exp(x + ix_1)\rho], \quad b > 0,$$

shows that estimate (8) is sharp.

Taking this opportunity, I express my gratitude to my scientific adviser M. A. Evgrafov for his constant attention and guidance.

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References

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- ³ M. A. Evgrafov, I. S. Arshon, DAN, **147**, No. 4 (1962).

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