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Abstract

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PROJECTIONS OF A TORSION-FREE METABELIAN GROUP

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Let us denote by φ a projection (structural isomorphism) of a group G onto a group G^φ. If H is a subgroup of G, then H^φ denotes its image in G^φ under the projection φ.

In the present note we consider projections of an arbitrary torsion-free metabelian group.

Lemma 1. *Let G be a torsion-free group containing at least two independent elements and with center Z different from the identity. Then every projection φ of the group G onto the group G^φ induces two one-to-one correspondences between the elements of these groups which, on every abelian subgroup $\{g, Z\}$, where g is an arbitrary element of G, are isomorphisms.*

Proof. Since the group G has no torsion and contains at least two independent elements, it has an abelian subgroup $\{a, Z\}$, $a \in G$, of rank not less than two. The projection φ induces a projection between the subgroups $\{a, Z\}$ and $\{\{a\}^\varphi, Z^\varphi\}$, where Z^φ is the center in G^φ ([1,2]). This projection is induced by two isomorphisms between the elements of the subgroups under consideration ([3]). Fix one of them. Thereby an isomorphism between the elements of Z and Z^φ is also fixed. The projection φ maps every abelian subgroup $\{g, Z\}$, where g is an arbitrary element of G, onto the abelian subgroup $\{\{g\}^\varphi, Z^\varphi\}$ of G^φ. In this case, to the fixed isomorphism between the elements of Z and Z^φ there corresponds one isomorphism between the elements of the subgroups $\{g, Z\}$ and $\{\{g^\varphi\}, Z^\varphi\}$, which generates the structural isomorphism between these subgroups induced by φ ([3]). As a result, a one-to-one correspondence is established between the elements of the groups G and G^φ, which we shall also denote by φ: $g^\varphi = g'$, where g' is a generating element of the subgroup $\{g\}^\varphi$ in $\{\{g\}^\varphi, Z^\varphi\}$, uniquely determined by the chosen isomorphism between the elements of Z and Z^φ. Therefore the projection φ is induced by two one-to-one correspondences φ and φ_1 between the elements of the groups G and G^φ. These correspondences do not coincide on any element g of the group G: if $g^\varphi = g'$, then $g^{\varphi_1} = {g'}^{-1}$ ([3]).

Remark. In what follows we shall agree to regard one of the two possible one-to-one correspondences between the elements of structurally isomorphic groups G and G^φ, in accordance with Lemma 1, as already fixed. This correspondence will be denoted below by the same letter φ, and the element of G^φ corresponding under φ to the element a of G by a^φ.

Lemma 2. *Every torsion-free metabelian group G with two generators is a free metabelian group.*

Proof. It is immediately clear that the commutant K of the group G is a cyclic subgroup. The isolator $I(k)$ of the commutant K is also a cyclic subgroup ([4], p. 429). Consider the quotient group $G/I(K)$. It is abelian and cannot be cyclic, since $I(K)$ is contained in the center of the group G. Consequently, $G/I(K)$ is an abelian group with two generators. This means that every metabelian group with two generators, including a free one, has rational rank ([5]) equal to three. But every torsion-free metabelian group is a quotient group of a free metabelian group. Since, by virtue of this, the group G was not

would be free, its rational rank could not be equal to three. This proves the lemma.

Lemma 3. Let φ be a projection of the torsion-free metabelian group G onto the group G^φ. Then, for any two noncommuting elements a and b of it and their commutator $k = b^{-1}a^{-1}ba = (b, a)$, for some integer λ and exponents $|\varepsilon| = |\mu| = |\eta| = 1$ there hold the correspondences:

$$\{ab\}^\varphi = \{(a^\varphi)^\varepsilon (b^\varphi)^\mu (k^\varphi)^\lambda\},$$

$$\{k\}^\varphi = \{k^\varphi\}; \qquad k^\varphi = [(b^\varphi)^{-1}(a^\varphi)^{-1} b^\varphi a^\varphi]^\eta.$$

Proof is contained directly in Lemmas 2.1 and 2.2 of ([6]), since, by Lemma 3, the subgroup $\{a, b\}$ of the group G turns out to be free metabelian.

Theorem. Every projection φ of a torsion-free metabelian non-locally cyclic group G carries it onto a group G^φ isomorphic to it and is induced by either an isomorphism or an anti-isomorphism between the elements of the groups G and G^φ.

Proof. Let G be a torsion-free metabelian non-locally cyclic group. Then G contains at least two independent elements and has a center Z distinct from the identity. Consequently, between the elements of the group G and of each of its structurally isomorphic images G^φ one can establish (in accordance with the remark to Lemma 1) a certain one-to-one correspondence φ, inducing the projection under consideration of G onto G^φ. We shall prove that the correspondence φ is either an isomorphism or an anti-isomorphism. Indeed:

1) Let a and b be any two commuting elements of G, but not from the center Z. Then the subgroup $H = \{a, b, Z\}$ is abelian of rank not less than two. The projection φ, carrying the subgroup H onto $H^\varphi = \{a^\varphi, b^\varphi, Z^\varphi\}$, with a fixed isomorphism between the elements of Z and Z^φ, is induced by exactly one isomorphism between the elements of the subgroups H and H^φ. Hence,

$$(ab)^\varphi = a^\varphi b^\varphi.$$

2) Let a and b be any two noncommuting elements of G. Then under the projection φ, according to Lemma 3, one of eight possibilities is realized:

$$(ab)^\varphi = (a^\varphi)^\varepsilon (b^\varphi)^\mu (k^\varphi)^\lambda; \tag{1}$$

$$k^\varphi = [(b^\varphi)^{-1}(a^\varphi)^{-1} b^\varphi a^\varphi]^\eta; \qquad |\varepsilon| = |\mu| = |\eta| = 1. \tag{2}$$

We shall show that in fact all these possibilities reduce to two: either

$$(ab)^\varphi = a^\varphi b^\varphi,$$

or

$$(ab)^\varphi = a^\varphi b^\varphi.$$

3) Suppose first that $\varepsilon = \mu = 1$, i.e. that

$$(ab)^\varphi = a^\varphi b^\varphi (k^\varphi)^\lambda.$$

In this case, according to (2), either $k^\varphi = k'$, or $k^\varphi = (k')^{-1}$, where

$$k' = (b^\varphi)^{-1}(a^\varphi)^{-1} b^\varphi a^\varphi.$$

3,1). Suppose first that $k^\varphi = k'$. We shall show that in this case, for every natural s, the correspondence

$$(a^s b^s)^\varphi = (a^\varphi)^s (b^\varphi)^s (k')^{\lambda s}$$

holds. Indeed, first, the elements ab and ba commute: $ba = abk$. Secondly,

$$(ba)^\varphi = (abk)^\varphi = (ab)^\varphi k^\varphi = a^\varphi b^\varphi (k^\varphi)^\lambda k^\varphi = a^\varphi b^\varphi (k')^{\lambda+1} = b^\varphi a^\varphi (k')^\lambda.$$

Further,

$$(ab)(ba) = a^2 b^2 k.$$

Then

$$(a^2 b^2 k^2)^\varphi = ((ab)(ba))^\varphi = (ab)^\varphi (ba)^\varphi = a^\varphi b^\varphi (k')^\lambda b^\varphi a^\varphi (k')^\lambda = a^\varphi (b^\varphi)^2 a^\varphi (k')^{2\lambda} = (a^\varphi)^2 (b^\varphi)^2 (k')^2 (k')^{2\lambda}.$$

Hence

$$(a^2 b^2)^\varphi = (a^2 b^2 k^2 k^{-2})^\varphi = (a^2 b^2 k^2)^\varphi (k^{-2})^\varphi = (a^\varphi)^2 (b^\varphi)^2 (k')^{2\lambda}.$$

Suppose the correspondence already proved:

$$(a^{s-1} b^{s-1})^\varphi = (a^\varphi)^{s-1} (b^\varphi)^{s-1} (k')^{(s-1)\lambda}.$$

Then note first that the elements $a^{s-1} b^{s-1}$ and ba commute:

$$(a^{s-1} b^{s-1})(ba) = a^{s-1} b^s a = a^s b^s k = b a^s b^{s-1} k = (ba)(a^{s-1} b^{s-1}).$$

Therefore

$$(a^s b^s k^s)^\varphi = ((a^{s-1}b^{s-1})(ba))^\varphi = ((a^{s-1})^\varphi (b^{s-1})^\varphi (k')^{(s-1)\lambda})(b^\varphi a^\varphi {k'}^\lambda) = (a^\varphi)^s (b^\varphi)^s (k')^s (k')^{\lambda s}.$$

Consequently,

$$(a^s b^s)^\varphi = (a^s b^s k^s k^{-s})^\varphi = (a^\varphi)^s (b^\varphi)^s {k'}^s {k'}^{\lambda s} {k'}^{-s} = (a^\varphi)^s (b^\varphi)^s {k'}^{\lambda s}.$$

At the same time,

time, by Lemma 2.2 ([6]), applied to the metabelian subgroup $\{a^p, b^p\}$, and taking the last relation into account, we obtain

$$(a^s b^s)^\varphi = (a^\varphi)^s (b^\varphi)^s [(b^\varphi)^s, (a^\varphi)^s]^\nu = (a^\varphi)^s (b^\varphi)^s (k')^{\nu s^2},$$

where ν depends, in general, on s. Comparing the correspondences obtained for any natural s, we find that $s\lambda = s^2\nu$, or $\lambda = s\nu$. But then, if at least one ν is equal to 0, then $\lambda = 0$. If, however, all ν are nonzero, then from the fact that every natural s divides λ, it also follows that $\lambda = 0$. Thus, in the case under consideration $(ab)^\varphi = a^\varphi b^\varphi$.

3.2) Suppose now that $k^\varphi = {k'}^{-1}$. We shall show that in this case $(ab)^\varphi = b^\varphi a^\varphi$. Similarly to 3.1), we obtain:

$$(ba)^\varphi = (abk)^\varphi = (ab)^\varphi k^\varphi = (a^\varphi b^\varphi {k'}^\lambda){k'}^{-1} = b^\varphi a^\varphi {k'}^{\lambda-2};$$

$$(a^2 b^2 k^2)^\varphi = ((ab)(ba))^\varphi = (ab)^\varphi (ba)^\varphi = (a^\varphi b^\varphi {k'}^\lambda)(b^\varphi a^\varphi {k'}^{\lambda-2}) = (a^\varphi)^2 (b^\varphi)^2 {k'}^{2\lambda}.$$

Then

$$(a^2 b^2)^\varphi = (a^2 b^2 k^2 k^{-2})^\varphi = (a^2 b^2 k^2)^\varphi (k^\varphi)^{-2} = (a^\varphi)^2 (b^\varphi)^2 {k'}^{2\lambda} {k'}^2 = (a^\varphi)^2 (b^\varphi)^2 {k'}^{2(\lambda+1)}.$$

Suppose it has already been shown that

$$(a^{s-1} b^{s-1})^\varphi = (a^\varphi)^{s-1} (b^\varphi)^{s-1} {k'}^{(s-1)(\lambda+s-2)}.$$

Then

$$(a^s b^s k^s)^\varphi = (a^{s-1} b^{s-1} ba)^\varphi = (a^{s-1} b^{s-1})^\varphi (ba)^\varphi$$

$$= (a^\varphi)^{s-1} (b^\varphi)^{s-1} {k'}^{(s-1)(\lambda+s-2)} b^\varphi a^\varphi {k'}^{\lambda-2} = (a^\varphi)^{s-1} (b^\varphi)^s a^\varphi {k'}^{(s-1)(\lambda+s-2)} {k'}^{\lambda-2} = (a^\varphi)^s (b^\varphi)^s {k'}^{s(\lambda+s-2)};$$

$$(a^s b^s)^\varphi = ((a^s b^s k^s) b^{-s})^\varphi = (a^\varphi)^s (b^\varphi)^s {k'}^{s(\lambda+s-1)}.$$

On the other hand,

$$(a^s b^s)^\varphi = (a^\varphi)^s (b^\varphi)^s {k'}^{s^2\nu}.$$

Consequently, $s(\lambda + s - 1) = s^2\nu$, or $\lambda - 1 = s(\nu - 1)$ for any natural s. If, for at least one s, the number $\nu = 1$, then $\lambda = 1$. If all ν are different from 1, then every natural s divides $\lambda - 1$. Hence $\lambda - 1 = 0$, and $\lambda = 1$. But then $(ab)^\varphi = b^\varphi a^\varphi$. Thus, if in correspondence (1) the exponents $\varepsilon = \mu = 1$, then either $\lambda = 0$, or $\lambda = 1$, and hence either $(ab)^\varphi = a^\varphi b^\varphi$, or $(ab)^\varphi = b^\varphi a^\varphi$.

4) In an analogous way one verifies that the realization of each of the remaining three possibilities represented by relation (1), in combination with either of the two possibilities represented by relation (2), leads either to

$$(ab)^{\varphi} = (a^{\varphi})^{\varepsilon}(b^{\varphi})^{\mu},$$

or to

$$(ab)^{\varphi} = (b^{\varphi})^{\mu}(a^{\varphi})^{\varepsilon}.$$

However, taking into account that, for any elements g of G and z of Z,

$$(gz)^{\varphi} = g^{\varphi}z^{\varphi},$$

it is easy to show the impossibility of at least one of the equalities: $\varepsilon = -1$, $\mu = -1$.

5) Thus it has been proved that for any two elements a and b of G, under the structural isomorphism φ one of the two relations holds:

$$(ab)^{\varphi} = a^{\varphi}b^{\varphi} \quad \text{or} \quad (ab)^{\varphi} = b^{\varphi}a^{\varphi}.$$

In this case, as shown in ([7]), the correspondence φ turns out to be either an isomorphism or an anti-isomorphism.

6) Finally, it is obvious that if φ is an isomorphism (anti-isomorphism), then the second one-to-one correspondence φ_1 (see Lemma 1) between the elements of the groups G and G^{φ}, inducing the projectivity φ under consideration, is an anti-isomorphism (isomorphism). This proves the theorem.

The theorem proved gives a positive solution to Problem 22.2.1 from the survey ([8]) for metabelian groups.

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Note: Figure translations are in progress. See original paper for figures.