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Abstract

Full Text

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ON THE REMAINDER IN MERTENS' FORMULA

(Presented by Academician I. M. Vinogradov on 30 VI 1962)

In the note (^{2}) an improvement was given of the remainder term in the classical Mertens formula for a segment of the Euler product at the point one. Here it will be shown how, using essentially the same method with a small modification, the remainder term in Mertens' formula can be replaced by a quantity of order

$$\exp[-a(\ln x)^{3/5}],$$

where a is an absolute constant.

Instead of the simple product $\Pi(s, x)$ of the note (^{2}), consider the averaged product

$$\Pi_c(s,x) = \left[\prod_{n\le x}\prod_{p\le n}\left(1-\frac{1}{p^s}\right)^{-1}\right]^{1/x} \qquad (1)$$

under the condition that x is an integer. For it the following is true.

Theorem. Let $s = \sigma + it,\ |t| \le x,\ \sigma \ge 1 - \dfrac{\lambda}{(\ln x)^{2/3}}$; then in this domain the equality

$$\Pi_c^{-1}(s,x)\zeta(s)(s-1) = \frac{e^{-c}}{\ln x}\, e^{\omega_0(s,x)}(1+\theta(s,x)), \qquad (2)$$

holds, where

$$\omega_0(s,x) = \int_{L_0} \frac{x^{1-w}-1}{w-1}\,dw + \int_{L_1} \frac{x^{1-w}}{2-w}\,dw,$$

L_0 is the straight line segment joining the points s and $(1,0)$; L_1 is a straight ray issuing from the point s and not passing through the point $(2,0)$; $\theta(s,x)$ is a function analytic in s and having in the indicated domain the estimate

$$|\theta(s,x)| \ll \exp[-a(\ln x)^{3/5}].$$

Proof. The product (1) is transformed into the form

$$\Pi_c(s,x) = \prod_{p\le x}\left(1-\frac{1}{p^s}\right)^{-(1-p/x)}.$$

Let us find the logarithmic derivative of $\Pi_c(s,x)$:

$$\frac{\Pi_c'}{\Pi_c}(s,x) = -\sum_{p\le x}\frac{(1-p/x)}{p^s-1}.$$

The sum, with the aid of the operator

$$\frac{1}{2\pi i}\int_{1-iT}^{1+iT}\frac{Y^w}{w(w+1)}\,dw$$

is easily brought to the form

$$\frac{1}{2\pi i}\int_{1-iT}^{1+iT}\frac{x^w}{w(w+1)}\frac{\zeta'}{\zeta}(s+w)\,dw + \theta_1(s,x), \tag{3}$$

where $\theta_1(s,x)$ is an analytic function having the estimate

$$|\theta_1(s,x)| \ll x^{-\sigma+1/2}\ln x.$$

Next, applying to the integral (3) the classical contour-shifting technique, set out in detail in (^{3}), and the technique of the note (^{2}), we obtain the theorem.

Corollary. The relation

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right) = \frac{e^{-c}}{\ln x}\left(1+O\{\exp[-a(\ln x)^{3/5}]\}\right). \tag{4}$$

is valid.

Proof. Letting $s\to 1$ in (2), we obtain

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{1-p/x} = \frac{e^{-c}}{\ln x}\exp\left(\int_{L_1}\frac{x^{1-w}}{2-w}\,dw\right)\left(1+O\{\exp[-a(\ln x)^{3/5}]\}\right), \tag{5}$$

where L_1 is a ray issuing from the unit point. The integral over L_1 is transformed into the form

$$\int_{L_1} \frac{x^{1-w}}{2-w}\, dw = \frac{1}{x}\int_2^x \frac{dy}{\ln y} + O\left(\frac{\ln x}{x}\right). \tag{6}$$

On the other hand,

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{1-p/x} = \prod_{p\leqslant x}\left(1-\frac{1}{p}\right)\exp\left(\frac{\pi(x)}{x} + O\left(\frac{\ln x}{x}\right)\right). \tag{7}$$

Substituting (6) and (7) into (5), we obtain

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right) = \frac{e^{-c}}{\ln x}\exp\left(\frac{\pi(x) - \int_2^x \frac{dy}{\ln y}}{x}\right)\left(1 + O\{\exp[-a(\ln x)^{3/5}]\}\right).$$

But, as is known from the latest theorems of I. M. Vinogradov on the boundary of the zeros for $\zeta(s)$,

$$\pi(x) - \int_2^x \frac{dy}{\ln y} = O\left(x\exp[-a(\ln x)^{3/5}]\right).$$

Consequently, equality (4) is indeed valid.

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CITED LITERATURE

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[2] A. I. Vinogradov, *DAN*, **143**, No. 5 (1962).
[3] A. Ingham, *Distribution of Prime Numbers*, IL, 1936.

Note: Figure translations are in progress. See original paper for figures.

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