



Soviet-era science, translated into English

V. E. LYANTSE

1963

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196301.19277>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

V. E. LYANTSE

ON INVERSION FORMULAS GENERATED BY A SECOND-ORDER BOUNDARY-VALUE PROBLEM ON THE HALF-LINE, IN A CLASS OF UNBOUNDED FUNCTIONS

(Presented by Academician L. S. Pontryagin on 14 XII 1962)

Let l_θ be the boundary-value problem generated by the differential expression $l[y] = -y'' + p(x)y$, given on the half-line $R^+ = [0, \infty)$, and by the boundary condition $y'(0) = \theta y(0)$. We shall assume that the complex-valued function $p(x)$ satisfies the condition

$$\int_0^\infty e^{\varepsilon x} |p(x)| dx < \infty, \tag{1}$$

where ε is some positive number; θ is an arbitrary complex number. As is known, the problem l_θ generates the so-called l_θ -Fourier transform. If, for simplicity, one assumes that the function $f(x)$, $x \in R^+$, is finite, then the l_θ -Fourier transform of the function $f(x)$ is defined by the relation

$$\omega(f, \lambda) = \int_0^\infty f(x)\omega(x, \lambda) dx, \tag{2}$$

where $\omega(x, \lambda)$ is the solution of the equation $l[y] = \lambda y$ with the initial values $\omega(0, \lambda) = 1$, $\omega'_x(0, \lambda) = \theta$. In particular, for $p(x) \equiv 0$ and $\theta = 0$, the l_θ -Fourier transform coincides with the Fourier cosine transform. The theory of generalized functions made it possible to extend the Fourier cosine transform to arbitrary functions growing as rapidly as desired.

In the present paper, using the results obtained in works ⁽¹⁻³⁾, and assuming that condition (1) is fulfilled, we extend the general l_θ -Fourier transform to functions $f(x)$ allowing a certain exponential order of growth at infinity. The definition we introduce is natural in the sense that it coincides with the extension by continuity from the class of finite functions in the corresponding topology.

Let $y_1(x, s)$ be the solution of the differential equation $l[y] = s^2 y$ satisfying the condition $|y_1(x, s) - \exp ixs| \rightarrow 0$ as $x \rightarrow \infty$ (see ^(1,4)). Put $A(s) = y'_{1x}(0, s) - \theta y_1(0, s)$. As is known, the function $A(s)$ is holomorphic in the half-plane $\text{Im } s > -\varepsilon/2$, satisfies the asymptotic equality $A(s) = is[1 + o(1)]$, and therefore the equation $A(s) = 0$ has a finite number of roots s_1, \dots, s_r in the half-plane $\text{Im } s > 0$ and a finite number of real roots $\sigma_1, \dots, \sigma_\rho \neq 0$. The

numbers $\lambda_1 = s_1^2, \dots, \lambda_r = s_r^2$ are eigenvalues of the boundary-value problem l_θ : if m_k is the multiplicity of the root s_k , then the functions $\{(d/d\lambda)^j \omega(x, \lambda)\}_{\lambda=s_k^2}$, $j = 0, \dots, m_k - 1$, satisfy the boundary condition $y'(0) = \theta y(0)$ and not only belong to the space $L^2(R^+)$, but even decrease as $x \rightarrow \infty$ at least as fast as $\exp(-\varepsilon_0 x)$, where

$$\varepsilon_0 = \min(\varepsilon/2, \varepsilon_1), \quad (3)$$

with ε_1 the distance from the real axis to the nonreal roots of the function $A(s)$. The numbers $\tilde{\lambda}_1 = \sigma_1^2, \dots, \tilde{\lambda}_p = \sigma_p^2$ are called spectral singularities of the boundary-value problem l_θ ; the corresponding

the “eigenfunctions and associated functions” : $\{(d/d\lambda)^j \omega(x, \lambda)\}_{\lambda=\tilde{\lambda}_k}$ do not belong to the space $L^2(R^+)$.

Denote by F the class of functions $f(x)$, $x \in R^+$, for each of which there exists a number $\eta = \eta(f) < \varepsilon_0$ such that

$$\|f\|_{-\eta} = \left\{ \int_0^\infty e^{-\eta x} |f(x)|^2 dx \right\}^{1/2} < \infty. \quad (4)$$

Of functions of the class F one may say that they “increase” more slowly than $\exp \varepsilon_0 x$. Below we introduce the notion of the l_θ -Fourier transform for functions $f(x)$ of the class F . Since these transforms will be generalized functions, one must first specify the corresponding basic space. To this end, for every infinitely differentiable function $\psi(x)$, $x \in R = (-\infty, \infty)$, set

$$\|\psi\|_{\eta q} = \sup_{x \in R} |e^{\eta|x|} \psi^{(q)}(x)| \quad (5)$$

and denote by Ψ the linear topological space consisting of functions ψ for which

$$\|\psi\|_{\eta q} < \infty \quad \text{for all } \eta < \varepsilon_0 \text{ and } q = 0, 1, \dots \quad (6)$$

with the natural linear operations and topology determined by the system of norms (6). It is not difficult to see that Ψ is a nuclear countably Hilbert space (see (5)). Further, for an arbitrary function $\tilde{\psi}(s)$ of the complex variable $s = \sigma + i\tau$, holomorphic in the strip $|\tau| < \varepsilon_0$, set

$$\|\tilde{\psi}\|_{\eta q} = \sup_{|\tau| < \eta} |s^q \tilde{\psi}(s)| \quad (7)$$

and denote by $\tilde{\Psi}$ the linear topological space corresponding (as above) to the relations

$$\|\tilde{\psi}\|_{\eta q} < \infty \quad \text{for all } \eta < \varepsilon_0, \quad q = 0, 1, \dots \quad (8)$$

For each function $\psi \in \Psi$ and each function $\tilde{\psi} \in \tilde{\Psi}$ set

$$E\psi(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ixs} dx, \quad |\operatorname{Im} s| < \varepsilon_0, \quad (9)$$

$$E\tilde{\psi}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\psi}(s) e^{-ixs} ds, \quad x \in R. \quad (10)$$

It can be proved that the Fourier operator E is a one-to-one and continuous mapping of the space Ψ onto the space $\tilde{\Psi}$ and of the space $\tilde{\Psi}$ onto the space Ψ . As usual (see (5)), by means of the relations

$$\langle \tilde{\psi}, E\psi \rangle = \langle E\tilde{\psi}, \psi \rangle, \quad \tilde{\psi} \in \tilde{\Psi}, \quad \langle \psi, E\tilde{g} \rangle = \langle E\psi, \tilde{g} \rangle, \quad \psi \in \Psi, \quad (11)$$

we extend the operator E to generalized functions $g \in \Psi'$ and generalized functions $\tilde{g} \in \tilde{\Psi}'$; here $\langle \psi, g \rangle$ ($\langle \tilde{\psi}, \tilde{g} \rangle$) is the value of the functional $g \in \Psi'$ ($\tilde{g} \in \tilde{\Psi}'$) on the element $\psi \in \Psi$ ($\tilde{\psi} \in \tilde{\Psi}$).

For an arbitrary function $f \in F$ set

$$Vf(x) = 0 \quad \text{for } x < 0, \quad Vf(x) = f(x) \quad \text{for } x > 0. \quad (12)$$

Identifying a regular functional with the function generating it pointwise, one may write the following inclusion: $Vf \in \Psi'$ for all $f \in F$. It turns out that, in the case when the function $f \in F$ is finite, for the integral (2) the formula is valid

$$s\omega(f, s^2) = \frac{1}{i} \sqrt{\frac{\pi}{2}} [A(S)E - A(-S)E^{-1}] V(1 + K)f(s); \quad (13)$$

in this case

$$Kf(t) = \int_0^t K(x, t) f(x) dx, \quad (14)$$

where $K(x, t)$ is a certain smooth kernel satisfying an inequality of the form

$$|K(x, t)| < Ce^{-\frac{\varepsilon}{2}(x+t)}, \quad (15)$$

so that the operator $1 + K$ maps the space F continuously onto itself; $A(S)$ and $A(-S)$ are the operators of multiplication by the functions $A(s)$ and $A(-s)$, respectively. We also note that the operator V is a continuous mapping of F into $\tilde{\Psi}'$, while the functions $A(s)$ and $A(-s)$ are multipliers in the space $\tilde{\Psi}$. Consequently, for each function $f \in F$, formula (13) defines a functional $s\omega(f, s^2) \in \tilde{\Psi}'$, and the mapping $f(x) \rightarrow s\omega(f, s^2)$ is a continuous transformation of F into $\tilde{\Psi}'$. Since $s\omega(f, s^2)$ is an even generalized function, after the change of variable $\lambda = s^2$ it becomes a generalized function $\omega(f, \lambda)$ defined on the half-axis $\lambda > 0$.

Definition. The l_θ -Fourier transform of a function $f \in F$ is the function ωf , which on the half-axis $\lambda > 0$ is equal to the generalized function $\omega(f, \lambda)$ defined by formula (13), and which, moreover, is holomorphic in some neighborhood of the eigenvalues $\lambda_1, \dots, \lambda_r$ and satisfies the relations

$$\{(d/d\lambda)^j \omega f(\lambda)\}_{\lambda=\lambda_k} = \int_0^\infty f(x) \{(d/d\lambda)^j \omega(x, \lambda)\}_{\lambda=\lambda_k} dx,$$

$$j = 0, \dots, m_k - 1, \quad k = 1, \dots, r^*.$$

Denote by μ_k the multiplicity of the real root σ_k of the equation $A(s) = 0$, $k = 1, \dots, \rho$.

Theorem 1. In order that the l_θ -Fourier transform ωf of a function $f \in F$ be equal to zero, it is necessary and sufficient that

$$f(x) = \sum_{k=1}^{\rho} \sum_{j=0}^{\mu_k-1} C_{kj} \left\{ \left(\frac{d}{d\lambda} \right)^j \omega(x, \lambda) \right\}_{\lambda=\tilde{\lambda}_k}; \quad (16)$$

here $\tilde{\lambda}_1 = \sigma_1^2, \dots, \tilde{\lambda}_\rho = \sigma_\rho^2$ are spectral singularities of the boundary-value problem l_θ , and C_{kj} are arbitrary constants.

Let us now consider the inversion of the l_θ -Fourier transform. It turns out that for every function $\tilde{g}(s)$, integrable on the axis $-\infty < s < \infty$, the following formula holds:

$$\int_{-\infty}^{\infty} s\omega(x, s^2) \tilde{g}(s) ds = i\sqrt{\frac{\pi}{2}} (1 + \tilde{K})V^{-1}I^+\tilde{\Gamma}_0\tilde{g}(x), \quad (17)$$

where $\tilde{\Gamma}_0 = E^{-1}A(-S) - EA(S)$, $I^+h(x) = 0$ for $x < 0$, and $I^+h(x) = h(x)$ for $x < 0^{**}$. $\tilde{K}f(x) = \int_x^\infty K(x, t)f(t) dt$, where $K(x, t)$ is the same kernel as above; the operator V is defined by relations (12). Denote by \mathfrak{D} the set of all those generalized functions $\tilde{g} \in \tilde{\Psi}'$ for which $V^{-1}I^+\tilde{\Gamma}_0\tilde{g} \in F$. It is clear that the

latter inclusion is possible only in the case when $\tilde{\Gamma}_0 \tilde{g}$ is a functional regular on R^+ . For generalized functions $\tilde{g} \in \mathfrak{D}$ we take the right-hand side of relation (17) as the definition of the integral appearing on its left-hand side. Next denote by

$$\frac{1}{A(s)A(-s)} \tilde{g}(s)$$

an arbitrary solution $\tilde{u}(s)$ of the equation $A(s)A(-s)\tilde{u}(s) = \tilde{g}(s)$ (recall that the functions $A(s)$ and $A(-s)$ are multipliers in the spa-

* Since the function f satisfies relation (4), all these integrals converge.

** $h(x)$ is a function defined on the whole axis $-\infty < x < \infty$.

of $\tilde{\Psi}$). If the boundary-value problem l_θ has no spectral singularities, then the “part”

$$\frac{1}{A(s)A(-s)} \tilde{g}(s)$$

has only one value. In the opposite case, the values of this “part” are determined up to a term spanning a linear finite-dimensional manifold (cf. (6)).

Theorem 2. For every function $f \in F$, every value of the part

$$\frac{1}{A(s)A(-s)} \mathfrak{F}\omega(f, s^2)$$

belongs to the set \mathfrak{D} , and the inversion formula is valid:

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathfrak{F}\omega(x, s^2) \frac{1}{A(s)A(-s)} \mathfrak{F}\omega(f, s^2) ds + \quad (18)$$

$$+ \sum_{k=1}^r \left\{ \left(\frac{d}{d\lambda} \right)^{m_k-1} M_k(\lambda) \omega f(\lambda) \omega(x, \lambda) \right\}_{\lambda=\lambda_k} + \sum_{k=1}^{\rho} \sum_{j=0}^{\mu_k-1} C_{kj} \left\{ \left(\frac{d}{d\lambda} \right)^j \omega(x, \lambda) \right\}_{\lambda=\tilde{\lambda}_k};$$

where

$$M_k(\lambda) = \frac{(\lambda - \lambda_k)^{m_k} y_1(0, \sqrt{\lambda})}{(m_k - 1)! A(\sqrt{\lambda})}, \quad (19)$$

and C_{kj} are certain constants depending on which of the possible values of the part

$$\frac{1}{A(s)A(-s)} \mathfrak{F}\omega(f, s^2)$$

is substituted under the “integral” sign.

Suppose now that $f \in L^2(R^+)$; then for each value of the “part”

$$\frac{1}{A(s)A(-s)} \mathfrak{F}\omega(f, s^2)$$

there exists a unique system of values of the constants C_{kj} for which the sum

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \mathfrak{F}\omega(x, s^2) \frac{1}{A(s)A(-s)} \mathfrak{F}\omega(f, s^2) ds + \sum_{k=1}^{\rho} \sum_{j=0}^{\mu_k-1} C_{kj} \left\{ \left(\frac{d}{d\lambda} \right)^j \omega(x, \lambda) \right\}_{\lambda=\tilde{\lambda}_k} \quad (20)$$

is an element of the space $L^2(R^+)$. With such a choice of the values of the constants C_{kj} , the sum (20) does not depend on the choice of the value of the “part”

$$\frac{1}{A(s)A(-s)} \mathfrak{F}\omega(f, s^2).$$

The value of the sum (20) belonging to the space $L^2(R^+)$ will be called the regularized value of the divergent integral

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\omega(f, s^2)\omega(x, s^2)s^2}{A(s)A(-s)} ds$$

and will be denoted by the symbol

$$\frac{1}{\pi} \text{reg. val.} \int_0^{\infty} \frac{\omega f(\lambda)\omega(x, \lambda)\sqrt{\lambda}}{A(\sqrt{\lambda})A(-\sqrt{\lambda})} d\lambda. \quad (21)$$

Theorem 3. Every function $f \in L^2(R^+)$ is uniquely determined by its l_θ -Fourier transform ωf , and the inversion formula holds

$$f(x) = \frac{1}{\pi} \text{reg. val.} \int_0^{\infty} \frac{\omega f(\lambda)\omega(x, \lambda)\sqrt{\lambda}}{A(\sqrt{\lambda})A(-\sqrt{\lambda})} d\lambda + \sum_{k=1}^r \left\{ \left(\frac{d}{d\lambda} \right)^{m_k-1} M_k(\lambda) \omega f(\lambda) \omega(x, \lambda) \right\}_{\lambda=\lambda_k}. \quad (22)$$

Lviv Polytechnic Institute

Received
11 XII 1962

References

1. M. A. Naimark, *Tr. Mosk. matem. obshch.*, **3**, 181 (1954).
2. B. Ya. Levin, *DAN*, **106**, No. 2, 187 (1956).
3. V. E. Lyantse, *DAN*, **149**, No. 2 (1963).
4. Z. S. Agranovich, V. A. Marchenko, *The Inverse Problem of Scattering Theory*, Kharkov, 1960.
5. I. M. Gel' fand, G. E. Shilov, N. Ya. Vilenkin, *Generalized Functions*, vols. 1-4, Moscow, 1958-1961.
6. L. Schwartz, *Théorie des distributions*, Paris, 1950.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.