MATHEMATICS

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Abstract

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ON THE REGULARIZATION OF ILL-POSED PROBLEMS

Numerous practically important problems lead to ill-posed problems, such as, for example, Fredholm integral equations of the first kind, the Cauchy problem for an elliptic equation, the problem of analytic continuation, etc. Ill-posed problems have recently attracted much attention (see, for example, ([1]), where the literature on this question is given).

The problem R of determining a function $z(s) \in Z$ from a given function $u(x) \in U$: $z(s) = R[s, u(x)]$ (where Z and U are the corresponding functional spaces) is called **well-posed** if:

1°. To every function $u(x) \in U$ there corresponds a solution $z(s)$ of the problem.
2°. The solution $z(s)$ is uniquely determined by the data $u(x)$.
3°. The solution $z(s)$ of the problem depends continuously on $u(x)$ in the metrics of Z and U.

Let an ill-posed problem $z = R[s, u]$ be given. As an illustrative example we shall take the Fredholm equation of the first kind

$$A[x, z(s)] = \int_a^b K(x, s) z(s)\, ds = u(x), \qquad c \le x \le d. \tag{1}$$

Not every function $\bar{u}(x)$ corresponds to a solution of the problem $\bar{z}(s)$.

Let us consider the following question concerning the solution of ill-posed problems.

It is known that to some function $\bar{u}(x)$ there corresponds a solution of the problem $\bar{z}(s) = R[s, \bar{u}(x)]$; suppose that a function $\tilde{u}(x)$ is given, an approximation to $\bar{u}(x)$ with known accuracy δ: $\|\bar{u}(x) - \tilde{u}(x)\| < \delta$. Determine $\tilde{z}(s)$—an approximate value of $\bar{z}(s)$ with prescribed accuracy $\|\tilde{z}(s) - \bar{z}(s)\|_z \le \varepsilon$, provided that δ—the accuracy with which $\tilde{u}(x)$ is specified—is sufficiently small. In this case $\tilde{z}(s)$ is by no means required to be equal to $R[s, \tilde{u}(x)]$, which may in general not exist.

We shall call an operator (or algorithm) $R_\delta[s, u(x)]$ **regularizing** if:

1°. $R_\delta[s, \tilde{u}(x)]$ is defined for all $\tilde{u} \in U$ and $\delta > 0$.

2°. If for $\bar{u}(x)$ there exists $\bar{z}(s) = R[s, \bar{u}(x)]$, then for every ε there exists a $\delta(\varepsilon, \bar{z})$ such that, if $\|\bar{u}(x) - \tilde{u}(x)\|_U < \delta$, then $\|\tilde{z}_\delta(s) - \bar{z}(s)\|_Z \leq \varepsilon$, where $\tilde{z}_\delta(s) = R_\delta[s, \tilde{u}]$.

We shall call the problem $z(s) = R[s, u(x)]$ **regularizable** if it admits at least one regularizing algorithm.

It is obvious that if the problem $z = R[s, u]$ is well-posed, then it is regularizable, since, putting $R_\delta[s, u] = R[s, u]$ for any δ, we obtain a regularizing algorithm.

Depending on the norm of z, we may distinguish weak regularization, uniform regularization, and regularization of the n-th order of smoothness.

Regularizing algorithms provide a practical method for solving ill-posed problems.

In (^{2}) a uniformly regularizing algorithm is given for equations of the first kind. In the present article, for the same class of problems, regularizing algorithms of the n-th order of smoothness are presented.

1. Consider the integral equation of the first kind (1), and for simplicity suppose that the kernel is continuous and that for $\bar{u}(x) = 0$ there is only the unique solution $\bar{z}(\delta) \equiv 0$. Consider the smoothing functional

$$M_n^\alpha[z(s); \bar{u}(x)] = N[z(s); \bar{u}(x)] + \alpha \Omega^{(n)}[z], \tag{2}$$

where

$$N[z(s); \bar{u}(x)] = \int_c^d |A[x, z(s)] - \bar{u}(x)|^2 \, dx$$

and the regularizing functional

$$\Omega^{(n)}[z] = \int_a^b \left\{ \sum_{i=0}^{n+1} K_i(s) \left(z^{(i)}(s)\right)^2 \right\} ds,$$

where $K_i(s)$ are continuous functions, $K_i(s) \geq 0$.

Theorem 1. For any function $\bar{u}(x) \in L_2$ and any $\alpha > 0$, there exists a unique $2(n+1)$-times continuously differentiable function $z_n^\alpha(s)$ realizing the minimum of the smoothing functional $M_n^\alpha[z, \bar{u}(x)]$.

The function $z_n^\alpha(s)$ is determined by the Euler equation

$$L_n^\alpha[z] = \alpha \left\{ \sum_{i=0}^{n+1} (-1)^{i+1} \frac{d^i}{ds^i} \left(K_i(s) \frac{d^i z}{ds^i} \right) \right\} - \left\{ \int_a^b \bar{K}(s, \zeta) z(\zeta) \, d\zeta - \bar{b}(s) \right\} = 0 \tag{3}$$

with boundary conditions

$$\pi^l(s) = \left\{ \sum_{i=l+1}^{n+1} (-1)^{i-l-1} \left[K_i(s) z^i(s)\right]^{(i-l-1)} \right\}\Big|_{a,b} = 0 \quad (l = 1, \dots, n+1), \tag{4}$$

where

$$\bar{K}(s, \zeta) = \int_a^b K(\xi, s) K(\xi, \zeta)\, d\xi, \qquad \bar{b}(s) = \int_c^d K(\xi, s) \bar{u}(\xi)\, d\xi.$$

Using the Green function for the boundary-value problem

$$\tilde{L}_n[z] = \sum_{i=0}^{n+1} (-1)^{i+1} \frac{d^i}{ds^i} \left[K_i z^{(i)}\right] = f, \qquad \pi^l(a) = \pi^l(b) = 0, \quad l = 1, \dots, n+1,$$

equation (3) is transformed into a Fredholm equation of the second kind, which for $\alpha > 0$ has only the trivial solution, and this proves the existence of the function $z_n^\alpha(s)$.

Theorem 1′**.** For any function $\bar{u}(x) \in L_2$ and any $\alpha > 0$, there exists a function $z^\alpha_{(-1)}(s) \in L_2$ realizing the minimum of the functional $M^\alpha_{(-1)}[z, \bar{u}]$.

This function is determined as the solution of the equation

$$\alpha K_0(z) z(s) = \int_a^b \bar{K}(s, \xi) z(\xi)\, d\xi - \bar{b}(s), \tag{3*}$$

for which the corresponding homogeneous equation has only the trivial solution.

Theorem 2. If $\bar{u}(x) = A[x, \bar{z}(s)]$, $\bar{z} \in \bar{C}^{(n+1)}$, then for any $\varepsilon > 0$ and auxiliary numbers $0 < \gamma_1 < \gamma_2$ there exists such a $\delta(\varepsilon, \gamma_1, \gamma_2, \bar{z})$ that if: 1) $\|u_\delta(x) - \bar{u}(x)\|_{L_2} \leq \delta$, where $u_\delta(x) \in L_2$; 2) $\alpha = \alpha(\delta)$ has pos-

has order δ^2: $\gamma_1 \leqslant \delta^2/\bar{\alpha}(\delta) \leqslant \gamma_2$, then $\bar{z}^{\bar{\alpha}}_{\delta,n}(s)$, realizing the minimum of the functional $M_n^{\bar{\alpha}}[z, \tilde{u}_\delta(x)]$, are such that

$$\left|\bar{z}^{\bar{\alpha}}_{\delta,n}(s)^{(i)} - \bar{z}(s)^{(i)}\right| \leqslant \varepsilon, \qquad a \leqslant s \leqslant b, \qquad i = 0, \dots, n,$$

for $\delta < \delta_0(\varepsilon, \gamma_1, \gamma_2, \bar{z})$.

Theorem 2′**.** If $\bar{u}(x) = A[x, \bar{z}(s)]$, $\bar{z} \in L_2$, then for any ε and auxiliary numbers $0 < \gamma_1 \leqslant \gamma_2$ there exists such a $\delta_0(\varepsilon, \gamma_1, \gamma_2, \bar{z})$ that, if: 1) $\|\tilde{u}_\delta(x) - \bar{u}(x)\|_{L_2} \leqslant \delta$, where $\tilde{u}_\delta(x) \in L_2$; 2) $\bar{\alpha} = \bar{\alpha}(\delta)$ has order δ^2: $\gamma_1 \leqslant \delta^2/\bar{\alpha}(\delta) \leqslant \gamma_2$, then $\bar{z}^{\bar{\alpha}}_{\delta,n}(s)$, realizing the minimum of the functional $M_n^{\bar{\alpha}(\delta)}[z, \tilde{u}_\delta(x)]$, converges weakly to $\bar{z}(s)$.

From Theorems 1, 2 and $1'$, $2'$ it follows that the solution of the boundary-value problem (3), (4) constitutes a regularizing algorithm of the n-th order of smoothness if $\bar{z} \in C^{n+1}$, and the solution of equation (3^*) constitutes an algorithm of weak regularization for $\bar{z} \in L_2$.

Analogously (^{2}), all these results carry over to multidimensional problems and to operator equations of the first kind

$$A[x, z(s)] = \bar{u}(x),$$

where $A[x, z]$ is an operator acting from C to L_2, bounded in the sense that, for almost every x,

$$|A[x, z(s)]| \leqslant A(x)\|z\|_C, \qquad \int_c^d A^2(x)\,dx = A_0 < +\infty.$$

A finite-dimensional approximation of the problem under consideration is treated analogously (^{2}).

2. Let us consider a very important special case of the problem under consideration, when

$$\int_a^b K(x, s)\,z(s)\,ds = \bar{u}(x), \qquad a \leqslant x \leqslant b$$

(i.e., when $a = c,\ b = d$) and when for $K(x, s)$ there exists a kernel of half-order, i.e.

$$K(x, s) = \int_{c'}^{d'} \widehat{K}(\xi, x)\widehat{K}(\xi, s)\,d\xi.$$

In this case the algorithm for obtaining $z^\alpha(s)$ is simplified. The equation for determining $z^\alpha(s)$ can be written in the form

$$L_n^{(\alpha)}[z] = \alpha\widetilde{L}_n[z] - \left\{\int_a^b K(s, \xi)\,z(\xi)\,d\xi - \bar{u}(s)\right\} = 0 \tag{3'}$$

with the conditions

$$\pi^l(a) = \pi^l(b) = 0 \qquad (l = 1, \dots, n+1). \tag{4'}$$

Indeed, consider the function

$$v(\xi) = \int_a^b K(\xi, s)\, z(s)\, ds, \qquad c' \leqslant \xi \leqslant d'.$$

Let us pose the problem: regarding $v(\xi)$ as a given function, find $z(s)$. This problem is equivalent to the original one and determines the same function $z(s)$. Applying to this

problem, by the basic algorithm set forth above, we obtain for determining $z^x(s)$ the problem $(3^*), (4^*)$, determined directly by the specification of $K(x, s)$, $\bar{u}(x)$.

3. The special case under consideration (Sec. 2) includes the problem of continuation of a potential toward the perturbing masses, determined by the Poisson equation

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{h}{(x-s)^2 + h^2}\, z(s)\, ds = \bar{u}(x), \qquad c \leq x \leq d.$$

If $z(s)$ is different from zero only for $a \leq s \leq b$ and the domain (c, d) in which $\bar{u}(x)$ is specified contains (a, b), then one may use the special algorithm $(3'), (4')$. If, however, (c, d) does not contain (a, b), then one must use the general algorithm $(3), (4)$.

Analogous considerations apply both to the inverse problem of heat conduction of the usual type

$$\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{a^2 t}} e^{-(x-s)^2/4a^2 t} z(s)\, ds = u(x),$$

and to the inverse problem of the second type, corresponding to the problem of determining the historical climate (3).

The problem of analytic continuation from the arc L_1 to the contour L_2, determined by the equation

$$w(z) = \frac{1}{2\pi i}\int_{L_2} \frac{w(\zeta)}{\zeta - z}\, d\zeta \qquad (z \subset L_1,\ \zeta \subset L_2)$$

is solved with the aid of the basic algorithm.

Problems of optimal regulation lead to ill-posed variational problems, and the regularization method also finds application in solving these problems.

The methods set forth have been tested on electronic computers and have yielded very effective results.

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Note: Figure translations are in progress. See original paper for figures.

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